HAESE MATHEMATICS

Specialists in mathematics education

Mathematics

Analysis and Approaches SL 2

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for use with IB Diploma Programme

DISCUSSION

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array} \qquad \begin{array}{ccccccccccc} & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \end{array}$$

These alternative representations of Pascal's triangle allow us to deduce some properties of the binomial coefficient $\binom{n}{r}$.

For example:

- The values of the coefficients at the end of each row are 1, suggesting that $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for all $n \in \mathbb{N}$.
- The remaining values in each row are found by adding the two values diagonally above it, giving us **Pascal's Rule** $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.
- The symmetry of Pascal's triangle suggests that $\binom{n}{r} = \binom{n}{n-r}$ for all $r, n \in \mathbb{N}$, $r \leqslant n$.

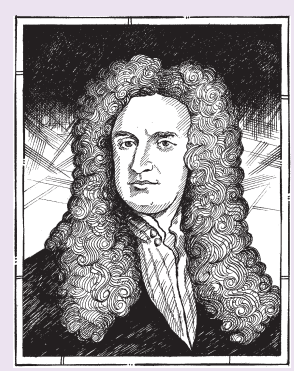
Can you explain, in the context of combinations, why these properties are true?

HISTORICAL NOTE — THE BINOMIAL THEOREM

The binomial theorem is one of the most important results in mathematics.

The process of multiplying out binomial terms dates back to the beginning of algebra. Mathematicians had noticed relationships between the coefficients for many centuries, and Pascal's triangle was certainly widely used long before Pascal.

Sir Isaac Newton discovered the binomial theorem in 1665, but he did not publish his results until much later. Newton was the first person to give a formula for the binomial coefficients. He did this because he wanted to go further. Newton's ground-breaking result included a generalisation of the binomial theorem to the case of $(a+b)^n$ where n is a rational number, such as $\frac{1}{2}$. In doing this, Newton was the first person to confidently use the exponential notation that we recognise today for both negative and fractional powers.



Sir Isaac Newton

REVIEW SET 1A

1 Express in factorial form:

a $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ **b** $10 \times 9 \times 8$

2 Simplify:

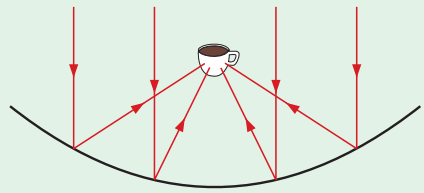
a $\dfrac{n!}{(n-2)!}$ **b** $\dfrac{n! + (n+1)!}{n!}$

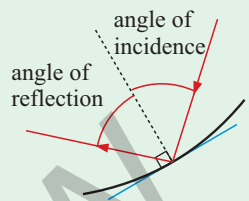
OPENING PROBLEM

Energy-conscious Misha wants to use solar energy to heat his cup of coffee. He has decided to build a reflecting surface to focus the sun's light on the cup.

He understands that the sun's rays will arrive parallel, and that each ray will bounce off the surface according to the law of reflection:

angle of incidence = angle of reflection





Things to think about:

a What *shape* should the surface have?

b Can we write a *formula* which defines the shape of the surface?

In this Chapter we will study **quadratic functions** and investigate their graphs which are called **parabolas**. There are many examples of parabolas in everyday life, including water fountains, bridges, and radio telescopes.

We will see how the curve Misha needs in the **Opening Problem** is actually a parabola, and how the **Opening Problem** relates to the geometric definition of a parabola.

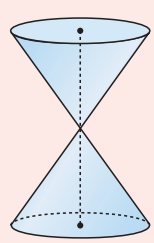
ACTIVITY 1 CONIC SECTIONS

A cone is *right-circular* if its apex is directly above the centre of the base.

Suppose we have two right-circular cones, and we place one upside-down on the first. Now suppose the cones are infinitely tall.

We call the resulting shape a **double inverted right-circular cone**.

When a double inverted right-circular cone is cut by a plane, 7 possible intersections may result, called **conic sections**:



- a point
- a line
- a line-pair
- a circle
- an ellipse
- a parabola
- a hyperbola

Click on the icon to explore the conic sections.

DEMO

You should observe how the parabola results when cutting the cone parallel to its slant edge.

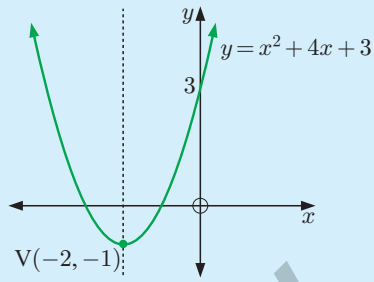
Example 6 **Self Tutor**

Write $y = x^2 + 4x + 3$ in the form $y = (x-h)^2 + k$ by "completing the square".
Hence sketch $y = x^2 + 4x + 3$, stating the coordinates of the vertex.

$y = x^2 + 4x + 3$
$\therefore\ y = x^2 + 4x + 2^2 + 3 - 2^2$
$\therefore\ y = (x+2)^2 - 1$

So, the axis of symmetry is $x = -2$
and the vertex is $(-2, -1)$.

When $x = 0$, $y = 3$
$\therefore$ the y-intercept is 3.

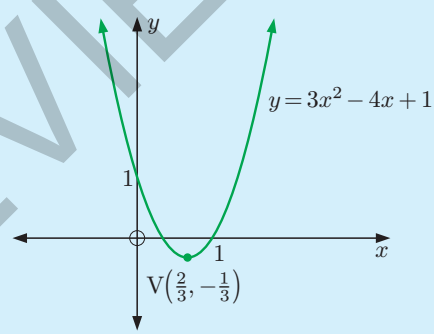


Example 7 **Self Tutor**

a Convert $y = 3x^2 - 4x + 1$ to the completed square form $y = a(x-h)^2 + k$.
b Hence write down the coordinates of the vertex, and sketch the quadratic.

a $y = 3x^2 - 4x + 1$
$= 3[x^2 - \frac{4}{3}x + \frac{1}{3}]$
$= 3[x^2 - 2(\frac{2}{3})x + (\frac{2}{3})^2 + \frac{1}{3} - (\frac{2}{3})^2]$
$= 3[(x - \frac{2}{3})^2 + \frac{3}{9} - \frac{4}{9}]$
$= 3[(x - \frac{2}{3})^2 - \frac{1}{9}]$
$= 3(x - \frac{2}{3})^2 - \frac{1}{3}$

b The vertex is $(\frac{2}{3}, -\frac{1}{3})$
and the y-intercept is 1.



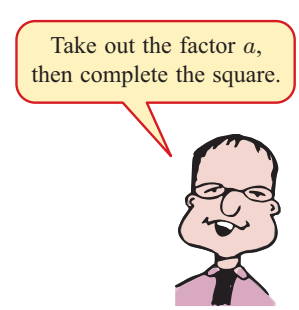
EXERCISE 2B.2

1 Write the following quadratics in the form $y = (x-h)^2 + k$ by "completing the square".
Hence sketch each function, stating the coordinates of the vertex.

a $y = x^2 - 2x + 3$ **b** $y = x^2 + 4x - 2$ **c** $y = x^2 - 4x$
d $y = x^2 + 3x$ **e** $y = x^2 + 5x - 2$ **f** $y = x^2 - 3x + 2$
g $y = x^2 - 6x + 5$ **h** $y = x^2 + 8x - 2$ **i** $y = x^2 - 5x + 1$

2 For each of the following quadratics:
i Write the quadratic in the completed square form $y = a(x-h)^2 + k$.
ii State the coordinates of the vertex.
iii Find the y-intercept.
iv Sketch the graph of the quadratic.

a $y = 2x^2 + 4x + 5$ **b** $y = 2x^2 - 8x + 3$
c $y = 2x^2 - 6x + 1$ **d** $y = 3x^2 - 6x + 5$
e $y = -x^2 + 4x + 2$ **f** $y = -2x^2 - 5x + 3$



11 Suppose $f(x) = \sqrt{1-x}$ and $g(x) = x^2$. Find:

a $(f \circ g)(x)$ **b** the domain and range of $(f \circ g)(x)$.

12 Suppose $f(x)$ and $g(x)$ are functions. $f(x)$ has domain D_f and range R_f. $g(x)$ has domain D_g and range R_g.

a Under what circumstance will $(f \circ g)(x)$ be defined?

b Assuming $(f \circ g)(x)$ is defined, find its domain.

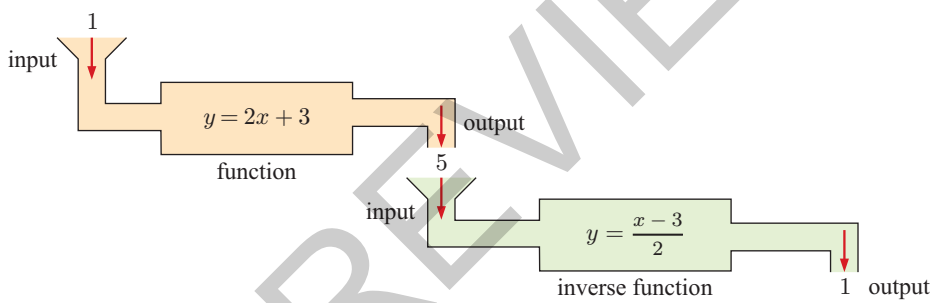
F INVERSE FUNCTIONS

The operations of $+$ and $-$, $\times$ and $\div$, are **inverse operations** as one "undoes" what the other does.

The function $y = 2x + 3$ can be "undone" by its *inverse* function $y = \dfrac{x-3}{2}$.

We can think of this as two machines. If the machines are inverses then the second machine *undoes* what the first machine does.

No matter what value of x enters the first machine, it is returned as the output from the second machine.



A function $y = f(x)$ *may or may not* have an inverse function. To understand which functions do have inverses, we need some more terminology.

ONE-TO-ONE AND MANY-TO-ONE FUNCTIONS

A **one-to-one** function is any function where:

- for each x there is only one value of y and
- for each y there is only one value of x.

Equivalently, a function is one-to-one if $f(a) = f(b)$ only when $a = b$.

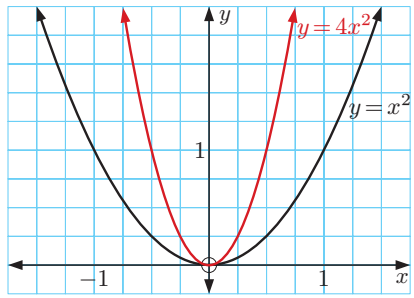
One-to-one functions satisfy both the **vertical line test** and the **horizontal line test**.

This means that:

- no vertical line can meet the graph more than once
- no horizontal line can meet the graph more than once.

In our study of quadratic functions, we saw that the coefficient a of x^2 controls the width of the parabola.

In the case of $f(x) = x^2$,
notice that $f(2x) = (2x)^2 = 4x^2$
and $4f(x) = 4x^2$



DISCUSSION

- In what ways could $y = x^2$ be *stretched* to form $y = 4x^2$?
- Will a transformation of the form $pf(x)$, $p > 0$ always be equivalent to a transformation of the form $f(qx)$, $q > 0$?

INVESTIGATION 2 STRETCHES

In this Investigation we consider transformations of the form $pf(x)$, $p > 0$, and $f(qx)$, $q > 0$.

What to do:

1 Let $f(x) = x + 2$.

GRAPHING PACKAGE

a Find, in simplest form:

i $3f(x)$ **ii** $\frac{1}{2}f(x)$ **iii** $5f(x)$

b Graph all four functions on the same set of axes.

c Which point is *invariant* under a transformation of the form $pf(x)$, $p > 0$?

An **invariant** point does not move.

d Copy and complete:

For the transformation $y = pf(x)$, each point becomes ..... times its previous distance from the x-axis.

2 Let $f(x) = x + 2$.

a Find, in simplest form:

i $f(2x)$ **ii** $f(\frac{1}{3}x)$ **iii** $f(4x)$

b Graph all four functions on the same set of axes.

c Which point is *invariant* under a transformation of the form $f(qx)$, $q > 0$?

d Copy and complete:

For the transformation $y = f(qx)$, each point becomes ..... times its previous distance from the y-axis.

INVESTIGATION 2 **THE "RULE OF 72"**

The "rule of 72" is used to estimate the time a quantity takes to double in value, given the rate at which the quantity grows.

Click on the icon to view this Investigation.

RULE OF 72

H LOGARITHMIC FUNCTIONS

We have seen that $\log_a a^x = a^{\log_a x} = x$.

Letting $f(x) = \log_a x$ and $g(x) = a^x$, we have $f \circ g = g \circ f = x$.

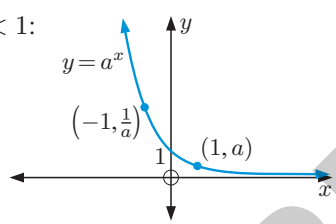
We can therefore say that the logarithmic function $\log_a x$ is the **inverse** of the exponential function a^x.

Algebraically, this has the effect that the logarithmic and exponential functions "undo" one another.

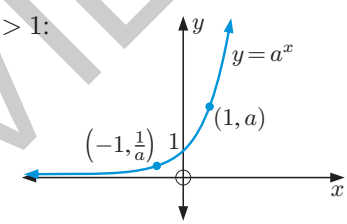
Geometrically, it means that the graph of $y = \log_a x$, $a > 0$, $a \neq 1$ is the *reflection* of the graph of $y = a^x$ in the line $y = x$.

We have seen previously the shape of the exponential function $y = a^x$ where $a > 0$, $a \neq 1$.

For $0 < a < 1$:



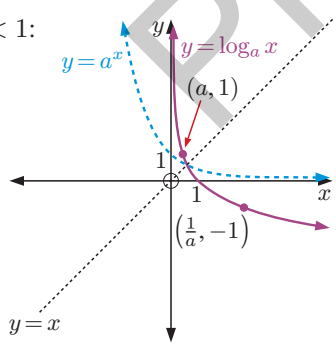
For $a > 1$:



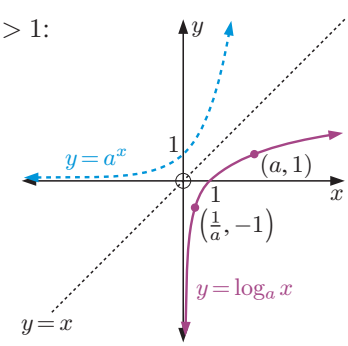
The horizontal asymptote for all of these functions is the x-axis $y = 0$.

By reflecting these graphs in the line $y = x$, we obtain the graphs for $y = \log_a x$.

For $0 < a < 1$:



For $a > 1$:



The **vertical asymptote** of $y = \log_a x$ is the y-axis $x = 0$.

For $0 < a < 1$: as $x \to \infty$, $y \to -\infty$
as $x \to 0^+$, $y \to \infty$

For $a > 1$: as $x \to \infty$, $y \to \infty$
as $x \to 0^+$, $y \to -\infty$

PROPERTIES OF $y = \log_a x$

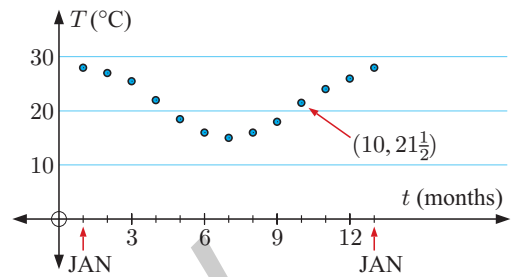
Since we can only find logarithms of positive numbers, the domain of $y = \log_a x$ is $\{x \mid x > 0\}$.

OBSERVING PERIODIC BEHAVIOUR

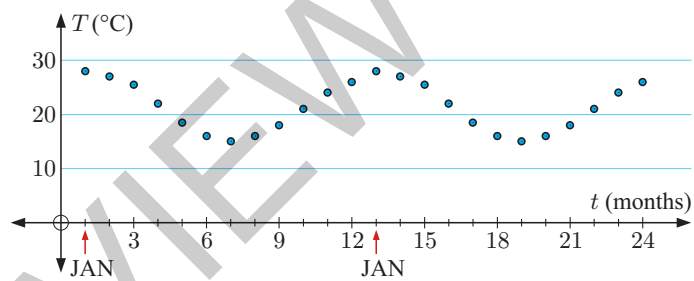
The table below shows the mean monthly maximum temperature for Cape Town, South Africa.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temperature T (°C)	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18	$21\frac{1}{2}$	24	26

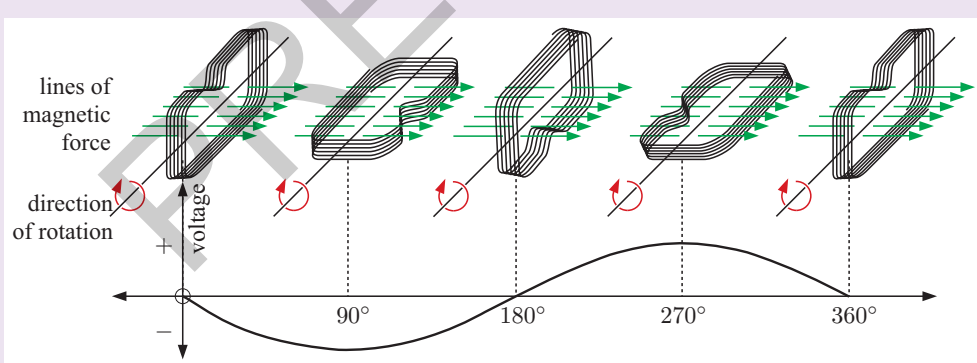
On the graph alongside we plot the temperature T on the vertical axis. We assign January as $t = 1$ month, February as $t = 2$ months, and so on for the 12 months of the year.



The temperature shows a variation from an average of 28°C in January through a range of values across the months. The cycle will approximately repeat itself for each subsequent 12 month period. By the end of the Chapter we will be able to establish a **periodic function** which approximately fits this set of points.



HISTORICAL NOTE



In 1831, **Michael Faraday** discovered that an electric current was generated by rotating a coil of wire at a constant speed through 360° in a magnetic field. The electric current produced showed a voltage which varied between positive and negative values in a periodic function called a **sine wave**.

TERMINOLOGY USED TO DESCRIBE PERIODICITY

A **periodic function** is one which repeats itself over and over in a horizontal direction, in intervals of the same length. The **period** of a periodic function is the length of one repetition or cycle.

$f(x)$ is a periodic function with period p if $\boldsymbol{f(x+p) = f(x)}$ for all x, and p is the smallest positive value for this to be true.

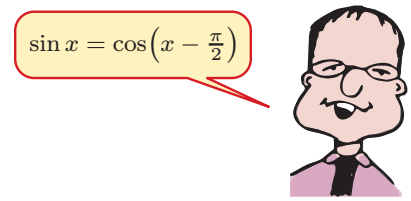
2 The data in the table shows the mean monthly temperatures for Christchurch, New Zealand.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14

a Find a cosine model for this data in the form $T \approx a\cos(b(t-c))+d$ without using technology. Let Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b Draw a scatter diagram of the data and sketch the graph of your model on the same set of axes.

c Use technology to check your answer to **a**.



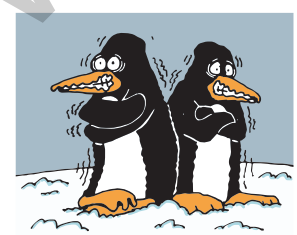
3 At the Mawson base in Antarctica, the mean monthly temperatures for the last 30 years are:

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	0	0	−4	−9	−14	−17	−18	−19	−17	−13	−6	−2

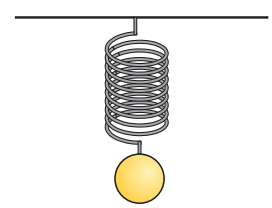
a Find a sine model for this data without using technology. Use Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b Draw a scatter diagram of the data and sketch the graph of your model on the same set of axes.

c How appropriate is the model?



4 An object is suspended from a spring. If the object is pulled below its resting position and then released, it will oscillate up and down. The data below shows the height of the object relative to its rest position, at different times.



Time (t seconds)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Height (H cm)	−15	−13	−7.5	0	7.5	13	15	13	7.5	0

Time (t seconds)	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Height (H cm)	−7.5	−13	−15	−13	−7.5	0	7.5	13	15	13	7.5

a Draw a scatter diagram of the data.

b Find a trigonometric function which models the height of the object over time.

c Use your model to predict the height of the object after 4.25 seconds.

d What do you think is unrealistic about this model? What would happen differently in reality?

RESEARCH

1 How accurately will a trigonometric function model the phases of the moon?

2 Are there any periodic phenomena which can be modelled by the *sum* of trigonometric functions?

D DEFINITIONS

THEORY OF KNOWLEDGE

In mathematics we clearly *define* terms so there is no misunderstanding of their exact meaning. For example:

- A **rational number** is a number which can be written in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$.
- An integer n is **even** if $n = 2k$ for some integer k.
- An integer n is **odd** if $n = 2k + 1$ for some integer k.

We can understand the need for specific definitions by considering integers and rational numbers:

- 2 is an integer, and is also a rational number since $2 = \frac{4}{2}$.
- $\frac{4}{2}$ is a rational number, and is also an integer since $\frac{4}{2} = 2$.
- $\frac{4}{3}$ is a rational number, but is *not* an integer.

1 Why is it important that mathematicians use the same definitions?

2 Words such as *similar*, *or*, *function*, *domain*, *range*, *period*, and *wave* are common words in English, but also have different or more specific mathematical definitions.
For each of these words, discuss the difference between their mathematical definition and their common use in English.

3 What is the difference between *equal*, *equivalent*, and *the same*? Why is it important to distinguish between these terms?

4 Are there any words which we use only in mathematics? What does this tell us about the nature of mathematics and the world around us?

Example 3 Self Tutor

Prove that $0.\overline{13} \in \mathbb{Q}$.

Let $x = 0.\overline{13} = 0.1313131313....$

$\Rightarrow \quad 100x = 13.13131313....$

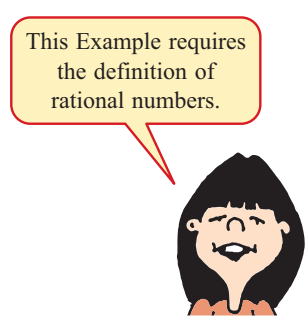
$\Rightarrow \quad 100x = 13 + x$

$\Rightarrow \quad 99x = 13$

$\Rightarrow \quad x = \frac{13}{99}$

$\Rightarrow \quad x \in \mathbb{Q}$

$\Rightarrow \quad 0.\overline{13} \in \mathbb{Q}$



Example 4 Self Tutor

Prove that the sum of any two rational numbers is also a rational number.

Proof:

Let x and y be two rational numbers.

By definition, there exists $p, q \in \mathbb{Z}$, $q \neq 0$ so that $x = \frac{p}{q}$.

By definition, there exists $r, s \in \mathbb{Z}$, $s \neq 0$ so that $y = \frac{r}{s}$.

EXERCISE 11D

1 F(3, 9) lies on the graph of $f(x) = x^2$. M also lies on the graph, and has x-coordinate $3 + h$.

a State the y-coordinate of M.

b Show that the gradient of the line segment [FM] is $6 + h$.

c *Hence* find the gradient of [FM] if M has coordinates:

i (4, 16) **ii** (3.5, 12.25)

iii (3.1, 9.61) **iv** (3.01, 9.0601)

d Use limits to find the gradient of the tangent to $f(x) = x^2$ at the point (3, 9).

2 **a** Find the gradient of the tangent to $f(x) = x^2$ at the point where:

i $x = 1$ **ii** $x = 4$

b Use **a** and other results from this Section to complete the table alongside for $f(x) = x^2$.

x-coordinate	*Gradient of tangent to* $f(x) = x^2$
1	
2	
3	
4	

c Predict the gradient of the tangent to $f(x) = x^2$ at the point where $x = a$.

3 Find the gradient of the tangent to:

a $f(x) = x^2 + x$ at the point (2, 6)

b $f(x) = x^3$ at the point where $x = 1$

c $f(x) = \dfrac{4}{x}$ at the point where $x = 2$

d $f(x) = x^4$ at the point where $x = 1$.

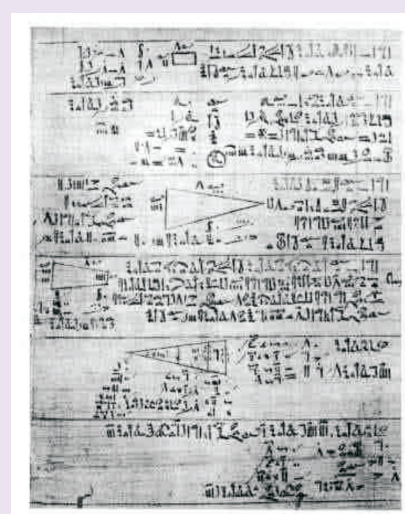
HISTORICAL NOTE

The word "calculus" is a Latin word referring to the small pebbles the ancient Romans used for counting.

The first known description of calculus is found on the **Egyptian Moscow papyrus** from about 1850 BC. Here, it was used to calculate areas and volumes.

Ancient Greek mathematicians such as **Democritus** and **Eudoxus** developed these ideas further by dividing objects into an infinite number of sections. This led to the study of **infinitesimals**, and allowed **Archimedes of Syracuse** to find the tangent to a curve other than a circle.

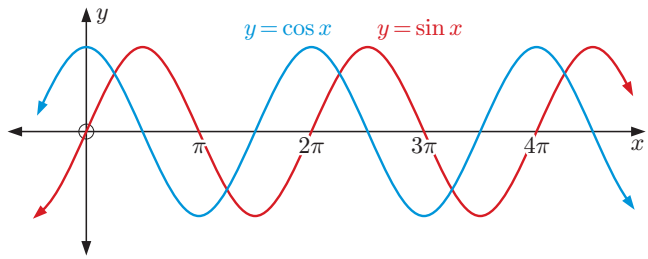
The methods of Archimedes were the foundation for modern calculus developed almost 2000 years later by mathematicians such as **Johann Bernoulli** and **Isaac Barrow**.



Egyptian Moscow papyrus

G DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

We have seen that the sine and cosine curves naturally arise from circular motion.



INVESTIGATION 7 DERIVATIVES OF $\sin x$ AND $\cos x$

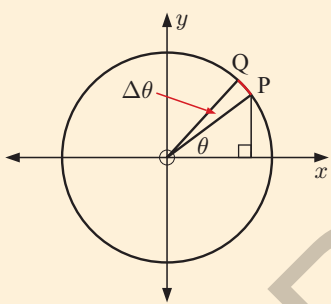
What to do:

1 Click on the icon to observe the graph of $y = \sin x$. A tangent with x-step of length 1 unit moves across the curve, and its y-step is translated onto the gradient graph. Predict the derivative of the function $y = \sin x$.

DERIVATIVES DEMO

2 Repeat the process in **1** for the graph of $y = \cos x$. Hence predict the derivative of the function $y = \cos x$.

3

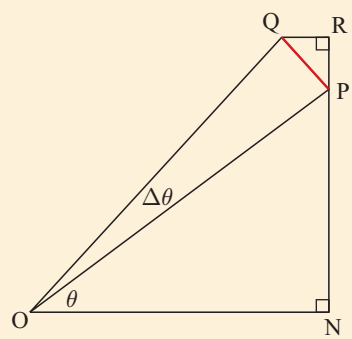


Suppose P and Q are points on the unit circle corresponding to angles θ and $\theta + \Delta\theta$ respectively from the positive x-axis.

a Explain why $\text{PR} = \sin(\theta + \Delta\theta) - \sin\theta$.

b If P and Q are close together then $\Delta\theta$ is very small. Explain why as Q approaches P:

- **i** the arc PQ resembles line segment [PQ]
- **ii** the length of the line segment $\text{PQ} \approx \Delta\theta$
- **iii** $\text{Q}\widehat{\text{P}}\text{O}$ approaches a right angle
- **iv** $\text{Q}\widehat{\text{P}}\text{R} \approx \theta$.



c Use right angled triangle trigonometry in $\triangle$QRP to show that $\cos\theta \approx \dfrac{\sin(\theta + \Delta\theta) - \sin\theta}{\Delta\theta}$.

Hence explain why in the limit as $\Delta\theta \to 0$, $\cos\theta = \dfrac{d}{d\theta}(\sin\theta)$.

From the **Investigation** you should have deduced that:

For x in radians: If $f(x) = \sin x$ then $f'(x) = \cos x$.
If $f(x) = \cos x$ then $f'(x) = -\sin x$.

12 Consider $f(x) = \ln(x(x-2))$.

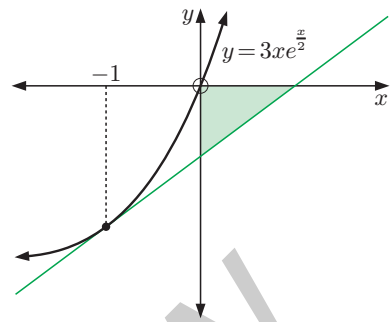
a State the domain of $f(x)$.

b Find $f'(x)$.

c Find the equation of the tangent to $y = f(x)$ at the point where $x = 3$.

13 Find the axes intercepts of the tangent to $y = x^2 e^x$ at $x = 1$.

14 Find the exact area of the shaded triangle.



15 Find the equation of the tangent to:

a $y = \sin x$ at the origin

b $y = \cos x$ at the point where $x = \frac{\pi}{6}$

c $y = \dfrac{1}{\sin 2x}$ at the point where $x = \frac{\pi}{4}$

d $y = \cos 2x + 3\sin x$ at the point where $x = \frac{\pi}{2}$.

16 Show that the curve with equation $y = \dfrac{\cos x}{1 + \sin x}$ does not have any horizontal tangents.

Example 4 **Self Tutor**

Find where the tangent to $y = x^3 + x + 2$ at $(1, 4)$ meets the curve again.

Let $f(x) = x^3 + x + 2$

$\therefore\ f'(x) = 3x^2 + 1$ and $\therefore\ f'(1) = 3 + 1 = 4$

$\therefore$ the equation of the tangent at $(1, 4)$ is $4x - y = 4(1) - 4$

or $y = 4x$.

The curve meets the tangent again when $x^3 + x + 2 = 4x$

$\therefore\ x^3 - 3x + 2 = 0$

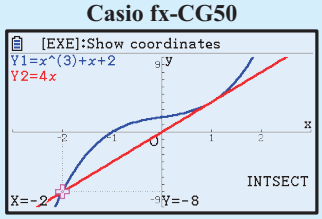
$\therefore\ (x-1)^2(x+2) = 0$

When $x = -2$, $y = (-2)^3 + (-2) + 2 = -8$

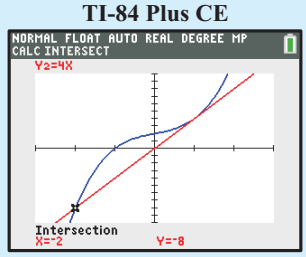
$\therefore$ the tangent meets the curve again at $(-2, -8)$.

$(x-1)^2$ must be a factor of $x^3 - 3x + 2 = 0$ since we are considering the *tangent* at $x = 1$.

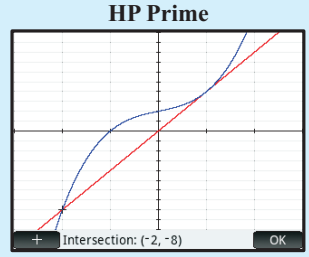
Casio fx-CG50



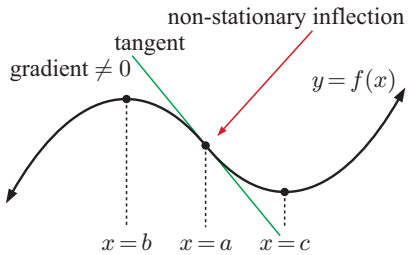
TI-84 Plus CE



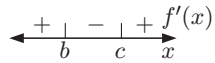
HP Prime



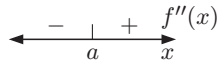
If the tangent at a point of inflection is *not* horizontal, then this is a **non-stationary inflection point**.



$f'(x)$ has sign diagram



$f''(x)$ has sign diagram



The tangent at the point of inflection, also called the **inflecting tangent**, crosses the curve at that point.

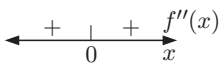
There is a **point of inflection** at $x = a$ if $f''(a) = 0$ **and** the sign of $f''(x)$ changes at $x = a$.

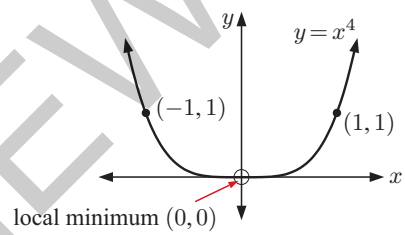
The point of inflection is a:

- **stationary inflection** if $f'(a) = 0$
- **non-stationary inflection** if $f'(a) \neq 0$.

Notice that if $f(x) = x^4$ then $f'(x) = 4x^3$
and $f''(x) = 12x^2$.

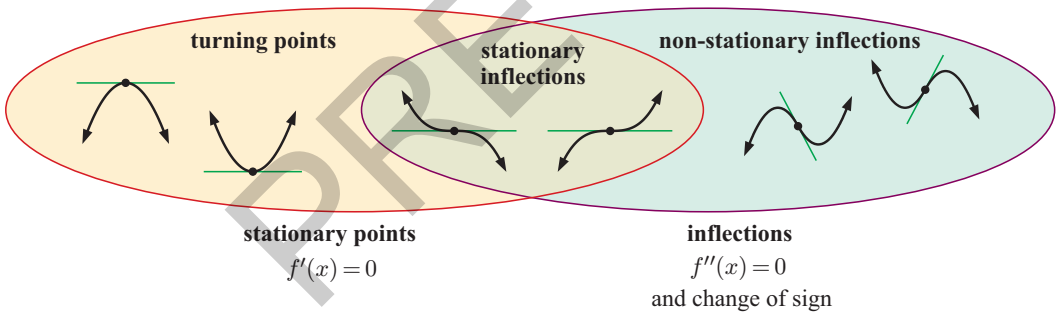
$f''(x)$ has sign diagram





Although $f''(0) = 0$ we do not have a point of inflection at $(0, 0)$ because the sign of $f''(x)$ does not change at $x = 0$.

SUMMARY



Click on the demo icon to examine some common functions for turning points, points of inflection, and intervals where the function is increasing, decreasing, and concave up or down.

DEMO

Example 15 Self Tutor

Consider $f(x) = 3x^4 - 16x^3 + 24x^2 - 9$.

a Find and classify all points where $f'(x) = 0$.

b Find and classify all points of inflection.

c Find intervals where the function is increasing or decreasing.

d Find intervals where the function is concave up or down.

e Sketch the function showing the features you have found.

OPENING PROBLEM

On the Indonesian coast, the depth of water t hours after midnight is given by $D = 9.3 + 6.8\cos(0.507t)$ metres.

Things to think about:

a What is the derivative function $\dfrac{dD}{dt}$ and what does it tell us?

b What is the depth of water at 8 am?

c Is the tide rising or falling at 8 am? Explain your answer.

d At what time(s) is the tide highest on this day? What is the maximum depth of water?

We have already seen that if $y = f(x)$ then $f'(x)$ or $\dfrac{dy}{dx}$ gives the gradient of the tangent to $y = f(x)$ for any value of x.

In this Chapter we consider some real-world applications of differential calculus, using derivatives to tell us how one variable changes relative to another.

A RATES OF CHANGE

There are countless examples in the real world where quantities vary with time, or with respect to some other variable.

For example:

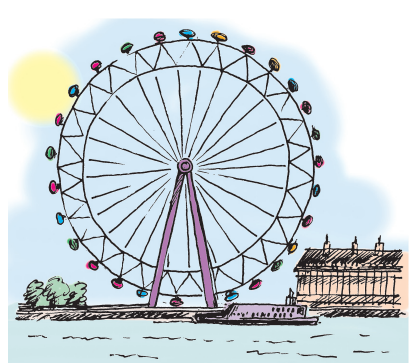
- temperature varies continuously
- the height of a tree varies as it grows
- the prices of stocks and shares vary with each day's trading.

$\dfrac{dy}{dx}$ gives the **rate of change in y with respect to x**.

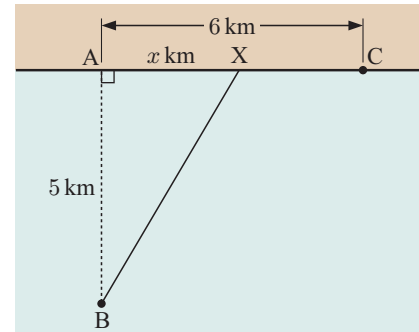
We can therefore use the derivative of a function to tell us the **rate** at which something is happening.

For example:

- $\dfrac{dH}{dt}$ or $H'(t)$ could be the instantaneous rate of ascent of a person in a Ferris wheel.
  It might have units metres per second or m s^{-1}.
- $\dfrac{dC}{dt}$ or $C'(t)$ could be a person's instantaneous rate of change in lung capacity.
  It might have units litres per second or L s^{-1}.



20 B is a boat 5 km out at sea from A. [AC] is a straight sandy beach, 6 km long. Peter can row the boat at 8 km h^{-1} and run along the beach at 17 km h^{-1}. Suppose Peter rows directly from B to point X on [AC] such that AX $= x$ km.



a Explain why $0 \leqslant x \leqslant 6$.

b Show that the *total time* Peter takes to row to X and then run along the beach to C, is given by

$$T = \frac{\sqrt{x^2+25}}{8} + \frac{6-x}{17} \text{ hours, } 0 \leqslant x \leqslant 6.$$

c Find x such that $\frac{dT}{dx} = 0$. Explain the significance of this value.

21 A mosquito flying with position M(x, y, z) is repelled by scent emitted from the origin O. At time t seconds, the coordinates of the mosquito are given by $x(t) = 3 - t^2$, $y(t) = 2 + \sqrt{t}$, and $z(t) = 2 - \sqrt{t}$, where all distance units are metres.

a Show that if the mosquito is D m from the origin at time t, then $D^2 = t^4 - 6t^2 + 2t + 17$.

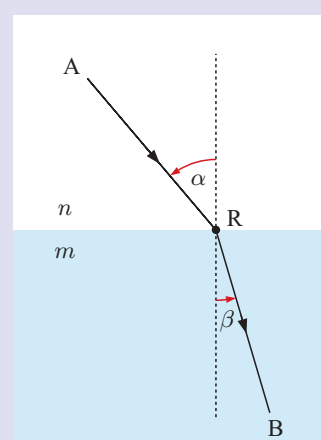
b *Hence* find the closest the mosquito came to the source of the repellent.

THEORY OF KNOWLEDGE

Snell's law states the relationship between the angles of incidence and refraction when a ray of light passes from one medium to another with different optical density. It was first discovered in 984 AD by the Persian scientist **Ibn Sahl**, who was studying the shape of lenses. However, it is named after **Willebrord Snellius**, who rediscovered it during the Renaissance. The law was published by **René Descartes** in his *Discourse on the Method* published in 1637.

Willebrord Snellius

In the figure alongside, a ray passes from A to B via point R. We suppose the refractive indices of the two media are n and m, the angle of incidence is α, and the angle of refraction is β.



Snell's law states that: $\boldsymbol{n \sin \alpha = m \sin \beta}$.

The law follows from Fermat's *principle of least time*, which says that a ray of light travelling between two points will take the path of least time.

1 Is optimisation a mathematical principle?

2 Is mathematics an intrinsic or natural part of other subjects?

8 Find:

a $\int \left(\sqrt{x} + \frac{1}{2}\cos x\right) dx$ **b** $\int (2e^x - 4\sin x)\, dx$ **c** $\int (3\cos x - \sin x)\, dx$

9 Find:

a $\int \left(x^2 - \frac{1}{x}\right)^2 dx$ **b** $\int \frac{x^2 - 4x + 2}{\sqrt{x}}\, dx$ **c** $\int \sqrt{x}(3x-1)^2\, dx$

DISCUSSION

In the rule $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1,$ why did we exclude the value $n = -1$?

C PARTICULAR VALUES

We can find the constant of integration c if we are given a particular value of the function.

Example 5 Self Tutor

Find $f(x)$ given that:

a $f'(x) = x^3 - 2x^2 + 3$ and $f(0) = 2$ **b** $f'(x) = 2\sin x - \sqrt{x}$ and $f(0) = 4$.

a

$$f'(x) = x^3 - 2x^2 + 3$$

$$\therefore\ f(x) = \int (x^3 - 2x^2 + 3)\, dx$$

$$\therefore\ f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + c$$

But $f(0) = 2$, so $c = 2$

Thus $f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + 2.$

b

$$f'(x) = 2\sin x - \sqrt{x}$$

$$\therefore\ f(x) = \int (2\sin x - x^{\frac{1}{2}})\, dx$$

$$\therefore\ f(x) = 2(-\cos x) - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\therefore\ f(x) = -2\cos x - \tfrac{2}{3}x^{\frac{3}{2}} + c$$

But $f(0) = 4$, so $-2\cos 0 - 0 + c = 4$

$\therefore\ c = 6$

Thus $f(x) = -2\cos x - \frac{2}{3}x^{\frac{3}{2}} + 6.$

EXERCISE 16C

1 Find $f(x)$ given that:

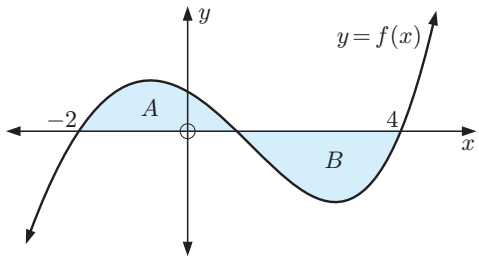
a $f'(x) = 2x - 1$ and $f(0) = 3$ **b** $f'(x) = 3x^2 + 2x$ and $f(2) = 5$

c $f'(x) = 2 + \frac{1}{\sqrt{x}}$ and $f(1) = 1$ **d** $f'(x) = x - \frac{2}{\sqrt{x}}$ and $f(1) = 2$

e $f'(x) = \sqrt{x} - 2$ and $f(4) = 0$ **f** $f'(x) = \frac{1}{x}$ and $f(e) = 2$.

2 A curve has gradient function $\frac{dy}{dx} = x - 2x^2$ and passes through $(2, 4)$. Find the equation of the curve.

4



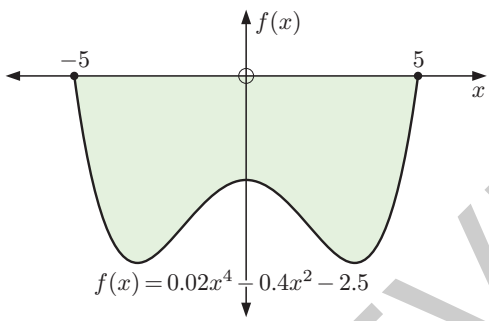
For the graph of $y = f(x)$ alongside, $\int_{-2}^{4} f(x)\,dx = -6$. Which region is larger, A or B? Explain your answer.

5 The area of the region bounded by $f(x) = -\frac{9}{x}$, the x-axis, $x = 3$, and $x = k$, is $9\ln 2$ units^2. Find the value of k.

6 **a** Sketch the graph of $y = 2\sin x + 1$ for $0 \leqslant x \leqslant 2\pi$.

b Find the area between the x-axis and the part of $y = 2\sin x + 1$ that is below the x-axis on $0 \leqslant x \leqslant 2\pi$.

7



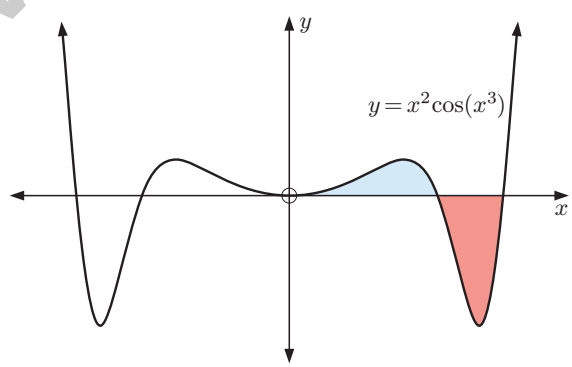
The cross-section of a roof gutter is defined by $f(x) = 0.02x^4 - 0.4x^2 - 2.5$, for $-5 \leqslant x \leqslant 5$ cm.

a Find the cross-sectional area of the gutter.

b The gutter is 20 m long. How much water can it hold in total?

8 **a** Find $\int x^2 \cos(x^3)\,dx$.

b Show that the red shaded region is twice as large as the blue shaded region.

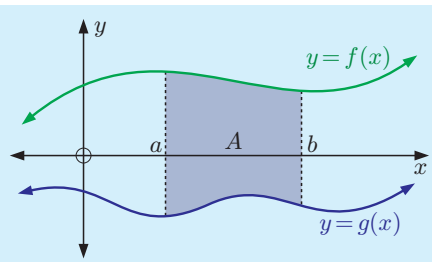


D THE AREA BETWEEN TWO FUNCTIONS

Consider two functions $f(x)$ and $g(x)$ where $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$.

The area between the two functions on the interval $a \leqslant x \leqslant b$ is given by

$$A = \int_a^b [f(x) - g(x)]\,dx.$$



THEORY OF KNOWLEDGE

Since the 1970s and 1980s, wage discrimination between men and women has been a topic of debate. During that time, Conway and Roberts[1] published a study which used linear regression to show that on average, women with the same qualifications as men were paid less. This would seem to imply that given the same salary, women would be more qualified. However, when the regression was applied the other way, the opposite conclusion was observed.

1 Can you explain why this occurred?

2 Are these two questions necessarily equivalent?
- "Given the same qualifications, do men and women earn the same wage?"
- "Given the same wage, do men and women have the same qualifications?"

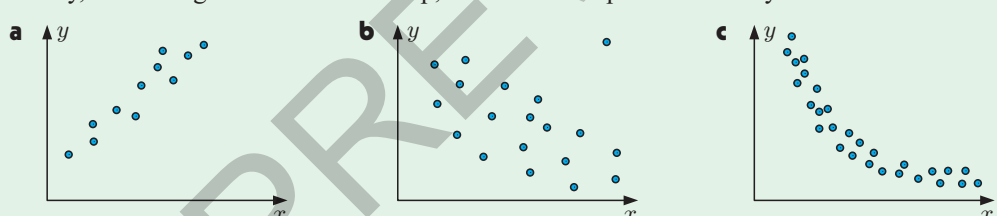
3 Is it necessary to consider both regression lines in order to conclude whether discrimination has occurred?

4 Should people be paid according to their qualifications, the job they do, or their capability in doing that job?

[1] Conway, Delores A. and Harry V. Roberts (1983). "Reverse Regression, Fairness, and Employment Discrimination". In: *Journal of Business & Economic Statistics* 1.1, pp. 75-85.

REVIEW SET 19A

1 For each scatter diagram, describe the relationship between the variables. Consider the direction, linearity, and strength of the relationship, as well as the presence of any outliers.



2 Kerry wants to investigate the relationship between the *water bill* and the *electricity bill* for the houses in her neighbourhood.

a Do you think the correlation between the variables is likely to be positive or negative? Explain your answer.

b Is there a causal relationship between the variables? Justify your answer.

3 Consider the data set alongside.

x	2	5	7	10	12	15
y	18	10	13	7	7	5

a Draw a scatter diagram for the data.

b Does the correlation between the variables appear to be positive or negative?

c Calculate Pearson's product-moment correlation coefficient r.

4 The table below shows the ticket and beverage sales for each day of a 12 day music festival:

Ticket sales ($\$x \times 1000$)	25	22	15	19	12	17	24	20	18	23	29	26
Beverage sales ($\$y \times 1000$)	9	7	4	8	3	4	8	10	7	7	9	8

a Draw a scatter diagram for the data.

Example 4 **Self Tutor**

Consider the magazine store from **Example 2**.

Find the expected number of magazines bought by each customer. Explain what this represents.

The probability table is:

x_i	1	2	3	4	5
p_i	0.23	0.38	0.21	0.13	0.05

In **Example 2** we found the mode and median for this distribution.

$$\begin{aligned} E(X) &= \sum_{i=1}^{n} x_i p_i \\ &= 1(0.23) + 2(0.38) + 3(0.21) + 4(0.13) + 5(0.05) \\ &= 2.39 \end{aligned}$$

In the long term, the average number of magazines purchased per customer is 2.39.

EXERCISE 20C.1

1 Find $E(X)$ for the following probability distributions:

a

x_i	1	2	3
p_i	0.4	0.5	0.1

b

x_i	0	1	2	3	4
p_i	0.1	0.2	0.15	0.2	0.35

c

x_i	0	2	5	10
p_i	0.2	0.35	0.27	0.18

d

x_i	10	15	30	60
p_i	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$

2 Consider the probability distribution alongside.

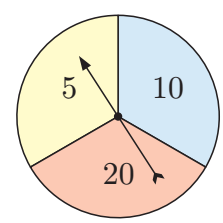
x	1	3	5
$P(X = x)$	$\frac{2}{5}$	a	$\frac{1}{10}$

a Find the value of a.

b Find the mode of the distribution.

c Find the mean μ of the distribution.

3 When the spinner alongside is spun, players are awarded the resulting number of points. In the long term, how many points can we expect to be awarded per spin?



4 When Ernie goes fishing, he catches 0, 1, 2, or 3 fish, with the probabilities shown.
On average, how many fish would you expect Ernie to catch on a fishing trip?

Number of fish	0	1	2	3
Probability	0.17	0.28	0.36	0.19

5 Each time Pam visits the library, she borrows either 1, 2, 3, 4, or 5 books, with the probabilities shown.

Number of books	1	2	3	4	5
Probability	0.16	0.15	a	0.28	0.16

a Find the value of a.

b Find the mode of the distribution.

c On average, how many books does Pam borrow per visit?

HISTORICAL NOTE

The French scholar **Pierre-Simon, Marquis de Laplace** (1749 - 1827) was the first to calculate $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. Using this result, it can be shown that $\int_{-\infty}^{\infty} f(x)\,dx = 1$ where $f(x)$ is the normal probability density function.

So, although we cannot calculate normal probabilities exactly with integration, we know and can prove that the total area under the normal curve is 1.

C THE STANDARD NORMAL DISTRIBUTION

Suppose a random variable X is normally distributed with mean μ and standard deviation σ.

For each value of x we can calculate a **z-score** using the algebraic transformation $z = \dfrac{x-\mu}{\sigma}$.

This algebraic transformation is known as the **Z-transformation**.

INVESTIGATION 3 — z-SCORES

In this Investigation we consider how the z-scores for a distribution are themselves distributed.

What to do:

1 Consider the x-values: 1, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 5, 6, 6, 7.

- **a** Draw a histogram of the x-values to check that the distribution is approximately normal.
- **b** Find the mean μ and standard deviation σ of the x-values.
- **c** Calculate the z-score for each x-value.
- **d** Find the mean and standard deviation of the z-scores.

2 Click on the icon to access a demo which randomly generates data values from a normal distribution with given mean and standard deviation. The z-score of each data value is calculated, and histograms of the original data and the z-scores are also shown.

DEMO

- **a** Generate samples using various values of μ and σ of your choosing.
- **b** Record the mean and standard deviation of the z-scores in a table like the one below.

x-values		*z-scores*	
Mean	*Standard deviation*	*Mean*	*Standard deviation*

- **c** How does the histogram of the z-scores generally compare with the histogram of the x-values?
- **d** What conclusions can you make about the *distribution* of the z-scores?