



**HAESE MATHEMATICS**

Specialists in mathematics education

# Mathematics

Applications and  
Interpretation SL

2



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for use with  
IB Diploma  
Programme

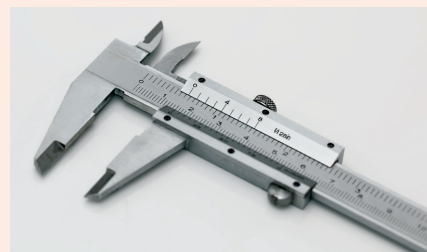
- 11 A rectangle is measured to be 6 cm by 8 cm. Find:
  - a the largest area it could have
  - b the smallest area it could have.
- 12 Find the boundary values for the actual area of a glass window measured as 42 cm by 26 cm.
- 13 The base of a triangle is measured as 9 cm and its height is measured as 8 cm. In what range of values does its actual area  $A$  lie?
- 14 Find the boundary values for the actual volume of a box measuring 4 cm by 8 cm by 6 cm.
- 15 Find the range of values in which the actual volume  $V$  of a house brick measuring 21.3 cm by 9.8 cm by 7.3 cm must lie.
- 16 A cylinder is measured to have radius 5 cm and height 15 cm. Find the boundary values for the cylinder's volume.
- 17 A cone is measured to have radius 8.4 cm and height 4.6 cm. Find the boundary values for the cone's volume.
- 18 Eko measures the diameter of a ball to be 18.2 cm. Do you expect the rounding in Eko's measurement to have more effect on a calculation of the ball's surface area, or a calculation of its volume? Explain your answer.
- 19 Rachel measures the base side lengths of a square-based pyramid to be 4.6 cm, and its height to be 5.2 cm. Find the boundary values for the pyramid's:
  - a volume
  - b surface area.

## RESEARCH

A **vernier scale** is used to measure the length of objects with a high degree of accuracy.

Research how vernier scales work.

## VERNIER SCALES



## D

## ABSOLUTE AND PERCENTAGE ERROR

Whenever we measure a quantity there is almost always a difference between our measurement and the actual value. We call this difference the **error**.

The *size* or *magnitude* of the error, whether the measured or estimated value is too high or too low, is called the **absolute error**.

If the actual or exact value is  $V_E$  and the approximate value is  $V_A$  then the

$$\text{absolute error} = |V_A - V_E|$$

Error is often expressed as a percentage of the exact value:

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$

In effect, an annuity is the reverse of a personal loan. The person loans the bank their money, and the bank makes regular repayments over time. As the balance in the annuity fund decreases, so does the interest it earns. However, the regular repayments remain the same.

Annuities are most commonly used as a source of regular income for people who have retired from work. Money saved during their working life is “rolled over” into an annuity fund. The regular withdrawals provide money for the person to live on.

## DISCUSSION

In some countries, all workers receive **superannuation** payments as part of their salaries. They do not have access to this money until they reach retirement age, so it is a form of compulsory saving.

Why might a government make superannuation compulsory?

When performing calculations with annuities:

- The person is depositing a lump-sum with the bank, so  $PV$  is negative.
- The person receives regular repayments, so  $PMT$  is positive.

### Example 3



Heather has just retired at age 65. She has \$900 000 in her savings fund. She “rolls over” the money into an annuity fund which returns 4% p.a. compounded monthly.

- If Heather withdraws \$5400 per month to live on, how long will it take for the money in the fund to run out?
- If Heather wants the money to last until she is 90 years old, how much can she afford to withdraw each month?

- We need to find how long it will take for the future value to fall to \$0.

$$I\% = 4, \quad PV = -900\,000, \quad PMT = 5400, \quad FV = 0, \\ P/Y = 12, \quad C/Y = 12 \\ \therefore N \approx 243.7$$

It will take 244 months (or 20 years 4 months) for the money in the fund to run out.

|                   |    |             |
|-------------------|----|-------------|
| Norm1             |    | End         |
| Compound Interest |    |             |
| n                 | =  | 243.6843051 |
| I%                | =  | 4           |
| PV                | =  | -900000     |
| PMT               | =  | 5400        |
| FV                | =  | 0           |
| P/Y               | =  | 12          |
| n                 | I% | PV          |
| PMT               | FV | AMORTIZ     |

- Heather wants the money to last  $90 - 65 = 25$  years.  
 $N = 25 \times 12 = 300$ ,  $I\% = 4$ ,  $PV = -900\,000$ ,  $FV = 0$ ,  
 $P/Y = 12$ ,  $C/Y = 12$   
 $\therefore PMT \approx 4750.53$   
 Heather can afford to withdraw \$4750.53 each month.

|                   |    |             |
|-------------------|----|-------------|
| Norm1             |    | End         |
| Compound Interest |    |             |
| n                 | =  | 300         |
| I%                | =  | 4           |
| PV                | =  | -900000     |
| PMT               | =  | 4750.531563 |
| FV                | =  | 0           |
| P/Y               | =  | 12          |
| n                 | I% | PV          |
| PMT               | FV | AMORTIZ     |

## EXERCISE 2B

- Sue has achieved her target of \$700 000 in her savings fund for her retirement at age 55. She rolls the money into an annuity account earning 4.85% p.a. compounded monthly.
  - How long will Sue’s money last if she withdraws \$4000 per month?
  - If Sue wants her money to last for 30 years, how much can she afford to withdraw per month?

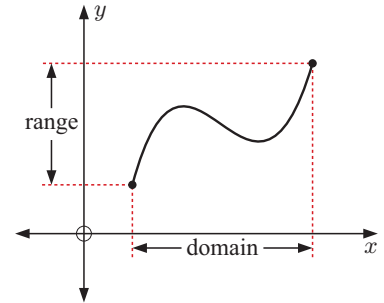
# C

## DOMAIN AND RANGE

We have seen that a relation is a set of points which connects two variables.

The **domain** of a relation is the set of values which the variable on the horizontal axis can take. This variable is usually  $x$ .

The **range** of a relation is the set of values which the variable on the vertical axis can take. This variable is usually  $y$ .



The domain and range of a relation can be described using **set notation**, **interval notation**, or a **number line graph**. For example:

| Set notation                            | Interval notation            | Number line graph | Meaning   |
|---|------------------------------|-------------------|---|
| $\{x \mid x \geq 3\}$                   | $x \geq 3$                   |                   | the set of all $x$ such that $x$ is greater than or equal to 3                      |
| $\{x \mid x < 2\}$                      | $x < 2$                      |                   | the set of all $x$ such that $x$ is less than 2                                     |
| $\{x \mid -2 < x \leq 1\}$              | $-2 < x \leq 1$              |                   | the set of all $x$ such that $x$ is between $-2$ and $1$ , including $1$            |
| $\{x \mid x \leq 0 \text{ or } x > 4\}$ | $x \leq 0 \text{ or } x > 4$ |                   | the set of all $x$ such that $x$ is less than or equal to $0$ , or greater than $4$ |

To find the domain and range of a function, we can observe its graph.

For example:

- (1) Consider again the car park charges in the **Opening Problem**.

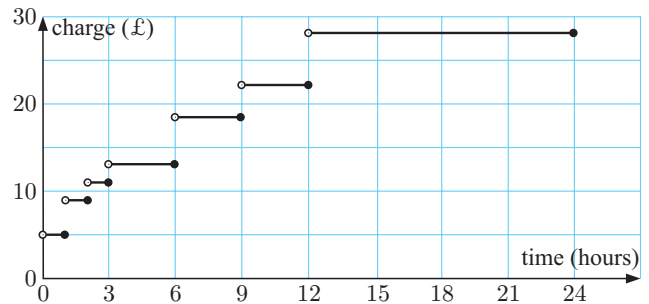
The function is defined for all times  $t$  such that  $0 < t \leq 24$ .

$\therefore$  the domain is  $\{t \mid 0 < t \leq 24\}$ .

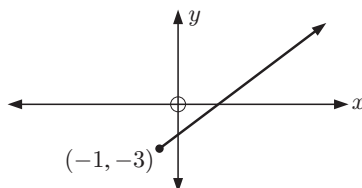
The possible charges are £5, £9, £11, £13, £18, £22, and £28.

$\therefore$  the range is

$\{5, 9, 11, 13, 18, 22, 28\}$ .



- (2)



All values of  $x \geq -1$  are included, so the domain is  $\{x \mid x \geq -1\}$ .

All values of  $y \geq -3$  are included, so the range is  $\{y \mid y \geq -3\}$ .

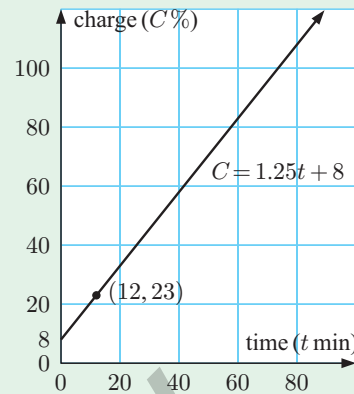
## OPENING PROBLEM

Briony's laptop had 8% charge left when she started to recharge it. 12 minutes later, the laptop had 23% charge.

Briony would like to know when the laptop will be fully charged. She constructs the model alongside to describe the laptop's charge after  $t$  minutes.

### Things to think about:

- a Can you explain how Briony constructed her model? What assumptions did Briony make? Do you think the assumptions are reasonable?
- b What range of values can each variable take? Has Briony accounted for this correctly?
- c According to Briony's model, how long will it take for the laptop to be fully charged?
- d The laptop actually took 79 minutes to be fully charged.
  - i Why do you think the actual time differed from the time predicted by Briony's model?
  - ii Do you think Briony's model was accurate enough to be useful?



We can apply mathematics to real-world problems to help understand the world around us.

A **mathematical model** is a simplified description of a real system using mathematical concepts and language. The process of developing this model is called **mathematical modelling**.

## A

## THE MODELLING CYCLE

Beyond basic problems of counting, most real-world problems are complicated. We generally cannot find an exact, perfect solution.

In mathematical modelling, we simplify the problem by making assumptions and removing less important details. This allows us to construct a mathematical description of the problem which is simple enough for us to work with, and which we can solve to approximate the real-world situation.

Mathematical models are developed using a **modelling cycle**:

- Step 1:* **Pose** a real-world problem. Make **assumptions** which simplify the problem without missing key features.
- Step 2:* **Develop** a model which represents the problem with mathematics. This may involve a formula or an equation. The model should consider constraints such as the range of possible values each variable can take in the real world.
- Step 3:* **Test** the model by comparing its predictions with known data. If the model is unsatisfactory, return to *Step 2*.
- Step 4:* **Reflect** on your model and **apply** it to your original problem, interpreting the solution in its real-world context.
- Step 5:* If appropriate, **extend** your model to make it more general or accurate as needed.

## E

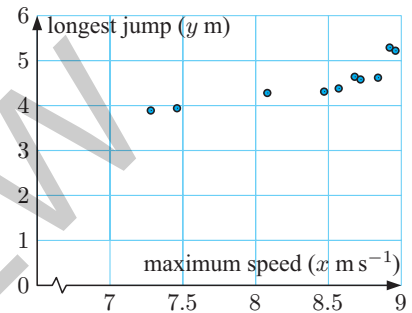
# SPEARMAN'S RANK CORRELATION COEFFICIENT

The table below shows data from 10 young athletes competing in a track and field competition. It includes the length of their longest jump, and the maximum speed they reached on their approach.

| <i>Athlete</i>   | A    | B    | C    | D    | E    | F    | G    | H    | I    | J    |
|--|------|------|------|------|------|------|------|------|------|------|
| <i>Maximum speed (<math>x \text{ m s}^{-1}</math>)</i> | 8.68 | 8.92 | 8.57 | 8.47 | 8.96 | 8.84 | 8.72 | 7.46 | 7.28 | 8.08 |
| <i>Longest jump (<math>y \text{ m}</math>)</i>         | 4.64 | 5.29 | 4.38 | 4.31 | 5.22 | 4.62 | 4.58 | 3.94 | 3.89 | 4.28 |

The scatter diagram of the data is shown alongside.

Pearson's correlation coefficient for the data  $\approx 0.864$ , which suggests a strong (but not perfect) positive correlation between the variables.

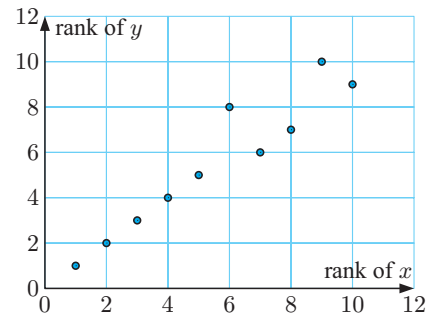


In a case like this, we may not be convinced that a linear model is appropriate for the data. We may therefore wish to focus just on the upward or downward *trend* of the data. One way to do this is to consider the relationship between each variable's **ranks** instead of the raw values.

For the above example, we can assign each  $x$ -value and each  $y$ -value a rank. A rank of 1 corresponds to the smallest value for the variable, rank 2 corresponds to the second smallest, and so on.

| <i>Athlete</i>   | A    | B    | C    | D    | E    | F    | G    | H    | I    | J    |
|--|------|------|------|------|------|------|------|------|------|------|
| <i>Maximum speed (<math>x \text{ m s}^{-1}</math>)</i> | 8.68 | 8.92 | 8.57 | 8.47 | 8.96 | 8.84 | 8.72 | 7.46 | 7.28 | 8.08 |
| <i>rank of <math>x</math></i>                          | 6    | 9    | 5    | 4    | 10   | 8    | 7    | 2    | 1    | 3    |
| <i>Longest jump (<math>y \text{ m}</math>)</i>         | 4.64 | 5.29 | 4.38 | 4.31 | 5.22 | 4.62 | 4.58 | 3.94 | 3.89 | 4.28 |
| <i>rank of <math>y</math></i>                          | 8    | 10   | 5    | 4    | 9    | 7    | 6    | 2    | 1    | 3    |

The scatter diagram for the ranked data is shown alongside. For this data we observe a more clear linear trend with Pearson's correlation coefficient  $\approx 0.952$ . This indicates a strong upward *but not necessarily linear* trend in the original data.



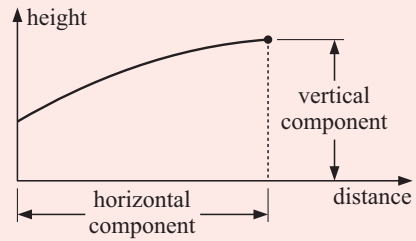
**Spearman's rank correlation coefficient** of a bivariate data set is defined as the Pearson product-moment correlation coefficient of the variables' **ranks**.

## ACTIVITY 3

## PROJECTILE MOTION

A **projectile** is an object upon which the only force acting is gravity. If we ignore air resistance, balls and missiles travel through the air with **projectile motion**.

When an object moves with projectile motion, it has both a **horizontal component** and a **vertical component**. It is often useful to consider each component as a separate function of time. This allows us to locate the object at any instant.

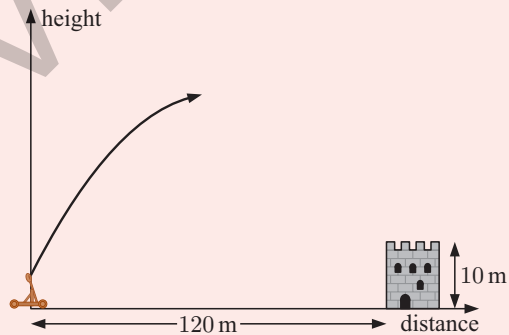
**What to do:**

- 1 Consider a ball thrown into the air. It moves with horizontal component  $x = 20t$  metres and vertical component  $y = -4.9t^2 + 14.7t + 1$  metres, where  $t$  is the time in seconds,  $t \geq 0$ .
  - a Find the position of the ball after:
    - i 1 second
    - ii 2 seconds.
  - b
    - i At what time was the ball at its highest point?
    - ii Find the maximum height reached by the ball.
    - iii How far had the ball travelled horizontally at this time?
  - c
    - i At what time did the ball hit the ground?
    - ii How far had the ball travelled horizontally when it hit the ground?

- 2 A stone is released from a catapult towards a castle. The castle is 120 m away, and is 10 m high.

The stone moves with horizontal component  $x = 30t$  metres and vertical component  $y = -4.9t^2 + 20t + 5$  metres, where  $t$  is the time in seconds,  $t \geq 0$ .

- a Find the position of the stone after 1 second.
- b Find the maximum height reached by the stone.
- c Will the stone fly over the castle, hit the castle, or land in front of the castle? Explain your answer.



- 3 Click on the icon to run a cannon simulation. Change the initial velocity and angle of trajectory, and observe the effect these have on the path of the cannonball. What angle of trajectory gives the greatest range for the cannon?

SIMULATION



## HISTORICAL NOTE

The theory of projectiles was developed in Europe in the 14th century, driven by the desire to improve guns and cannons. At that time, scientists were still using Aristotle's theory of motion which suggested that forces gave rise to momentum. This would mean that as soon as you stopped pushing something (even an object on wheels) it would stop moving.

- 7 The force between two positively charged spheres was measured at different distances. The results are recorded below.

|                   |     |      |      |      |      |      |
|-------------------|-----|------|------|------|------|------|
| Distance ( $d$ m) | 0.1 | 0.25 | 0.5  | 0.75 | 1    | 1.5  |
| Force ( $F$ N)    | 563 | 90.0 | 22.5 | 10.0 | 5.63 | 2.50 |

- Obtain a power model which best fits the data.
  - Estimate the strength of the force when the spheres are 0.4 m apart.
  - Find the distance between the spheres if the force between them is 650 N.
- 8 Donna had heard that the pressure of a constant volume of gas is directly proportional to its temperature.

To test this for herself, Donna observed how the pressure in a tube changed as the gas inside was heated. Her results are shown in the table:

|                                |      |      |      |      |      |      |      |
|--------------------------------|------|------|------|------|------|------|------|
| Temperature ( $T$ °C)          | 5    | 10   | 15   | 20   | 25   | 30   | 35   |
| Pressure ( $P \times 10^5$ Pa) | 3.22 | 3.28 | 3.33 | 3.39 | 3.45 | 3.51 | 3.57 |

- Obtain the power model which best fits the data. Explain why  $P$  is *not* directly proportional to  $T$ .
- Donna's friend Penny suggests that the temperatures should be recorded in kelvin (K) rather than degrees Celsius.
  - Redraw the table with the temperatures recorded in kelvin.
  - Obtain the power model which best fits the data. Use the correlation coefficient to explain why this power model is a much better fit for the data.
  - Is it now reasonable to conclude that pressure and temperature are directly proportional? Explain your answer.

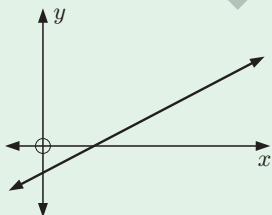
To convert temperatures from °C to K, add 273.15.



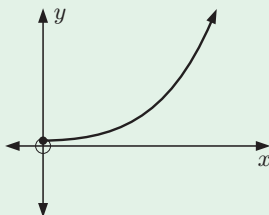
## REVIEW SET 7A

- 1 Which graph indicates that  $y$  is directly proportional to  $x$ ?

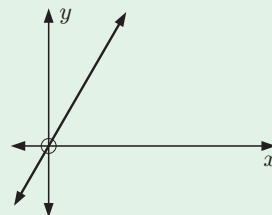
A



B



C



- 2 Suppose  $A$  is directly proportional to  $t$ . Explain what happens to:

a  $A$  when  $t$  is multiplied by 4

b  $t$  when  $A$  is increased by 5%.

- 3 A person's weight on the Moon is directly proportional to their weight on Earth. A person weighing 750 N on Earth weighs only 124 N on the Moon. Given that John weighs 640 N on Earth, find his weight on the Moon.



**Example 11****Self Tutor**

The magnitude of an earthquake which releases  $E$  joules of energy is given by  $M = \frac{2}{3} \log \left( \frac{E}{10^{4.8}} \right)$ .

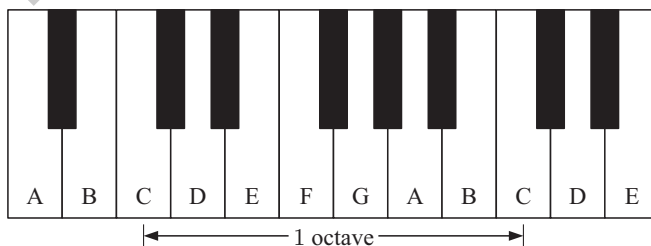
- a Find the magnitude of an earthquake which releases  $5.1 \times 10^{10}$  joules of energy.
- b How much energy is released by a magnitude 4.3 earthquake?

a When  $E = 5.1 \times 10^{10}$ ,  $M = \frac{2}{3} \log \left( \frac{5.1 \times 10^{10}}{10^{4.8}} \right)$   
 $\approx 3.94$

b When  $M = 4.3$ ,  $4.3 = \frac{2}{3} \log \left( \frac{E}{10^{4.8}} \right)$   
 $\therefore$  using technology,  $E \approx 1.78 \times 10^{11}$  joules.

Math Rad Norm1 d/c Real  
 Eq: 4.3 =  $\frac{2}{3} \log \frac{x}{10^{4.8}}$   
 $x = 1.77827941 \times 10^{11}$   
 Lft = 4.3  
 Rgt = 4.3  
 [REPEAT]

- 12 Use the formula in the Example above to find:
  - a the magnitude of an earthquake which releases  $6.2 \times 10^{13}$  joules of energy
  - b the energy released by a magnitude 5.1 earthquake.
- 13 In chemistry, the **pH** scale is used to measure acidity. The pH of a solution is given by  $\text{pH} = -\log C$ , where  $C$  is the concentration of  $\text{H}_3\text{O}^+$ . Find:
  - a the pH of a solution with  $\text{H}_3\text{O}^+$  concentration  $0.000\,234 \text{ mol L}^{-1}$
  - b the  $\text{H}_3\text{O}^+$  concentration in a solution with pH 6.2.
- 14 Musical notes are named according to the frequency of their sound waves. They are labelled with letters of the alphabet. A note which has *twice* the frequency of another is said to be one **octave** higher than it. So, one C is an octave below the next C.  
 A note  $n$  octaves above “Middle C” has frequency  $f$  Hz. The variables  $n$  and  $f$  are related by the equation  $n \approx 3.322 \log \left( \frac{f}{261.6} \right)$ .



- a Find the frequency of “Middle C”.
- b How many octaves above “Middle C” is a note with frequency 784 Hz?
- c Find the frequency of the note:
  - i 3 octaves above “Middle C”
  - ii 1 octave below “Middle C”.
- d There are 12 different notes in an octave. They are equally spaced on the logarithmic scale. Find the ratio of frequencies between two adjacent notes.

In the following **Investigation**, we will use technology to draw graphs related to  $y = \sin x$ .

**INVESTIGATION 1****FAMILIES OF TRIGONOMETRIC FUNCTIONS****What to do:**

- 1 a** Use the graphing package to graph on the same set of axes:

**i**  $y = \sin x$

**ii**  $y = 2 \sin x$

**iii**  $y = \frac{1}{2} \sin x$

**iv**  $y = -\sin x$

**v**  $y = -\frac{1}{3} \sin x$

**vi**  $y = -\frac{3}{2} \sin x$

- b** For graphs of the form  $y = a \sin x$ , comment on the significance of:

**i** the sign of  $a$

**ii** the size of  $a$ , or  $|a|$ .

- 2 a** Use the graphing package to graph on the same set of axes:

**i**  $y = \sin x$

**ii**  $y = \sin 2x$

**iii**  $y = \sin \frac{x}{2}$

**iv**  $y = \sin 3x$

- b** For graphs of the form  $y = \sin bx$ ,  $b > 0$ , what is the period of the graph?

- 3 a** Graph on the same set of axes:

**i**  $y = \sin x$

**ii**  $y = \sin x + 2$

**iii**  $y = \sin x - 2$

- b** What translation moves  $y = \sin x$  to  $y = \sin x + d$ ?

- c** What is the principal axis of  $y = \sin x + d$ ?

- 4** What sequence of transformations maps  $y = \sin x$  onto  $y = a \sin(bx) + d$ ?

GRAPHING PACKAGE



$|a|$  is the size of  $a$ , ignoring its sign.  
So,  $|2| = 2$  and  $|\frac{1}{3}| = \frac{1}{3}$ .



From the **Investigation** you should have observed the following properties about the **general sine function**  $y = a \sin(bx) + d$ :

$$y = a \sin(bx) + d$$

affects  
amplitude

affects  
period

affects  
vertical translation

- The amplitude is  $|a|$ .
- The period is  $\frac{360^\circ}{b}$  for  $b > 0$ .
- The principal axis is  $y = d$ .
- $y = a \sin(bx) + d$  is obtained from  $y = \sin x$  by a vertical stretch with scale factor  $a$  and a horizontal stretch with scale factor  $\frac{1}{b}$ , followed by a vertical translation of  $d$  units.

Click on the icon to obtain a demonstration for the general sine function.

The properties of the **general cosine function**  $y = a \cos(bx) + d$  are the same as those of the general sine function.

DEMO



**Example 12****Self Tutor**

At what point on the curve  $y = 2x^2 + 5x - 3$  does the tangent have gradient 13?

$$y = 2x^2 + 5x - 3$$

$$\therefore \frac{dy}{dx} = 4x + 5$$

$$\therefore \text{the tangent has gradient 13 when } 4x + 5 = 13$$

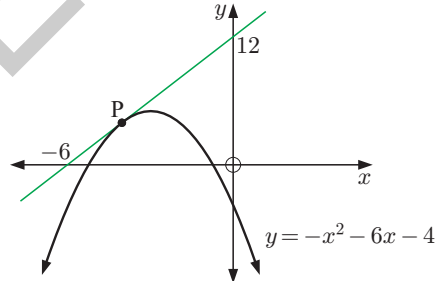
$$\therefore 4x = 8$$

$$\therefore x = 2$$

$$\text{When } x = 2, y = 2(2)^2 + 5(2) - 3 = 15$$

So, the tangent has gradient 13 at the point  $(2, 15)$ .

- 12** At what point on the graph of  $y = x^2 - 4x + 7$  does the tangent have gradient 2? Draw a diagram to illustrate your answer.
- 13** Find the coordinates of the point(s) on:
- a**  $y = x^2 + 5x + 1$  where the tangent has gradient 3
  - b**  $y = 3x^2 + 11x + 5$  where the tangent has gradient  $-7$
  - c**  $f(x) = 2x^{-2} + x$  where the tangent has gradient  $\frac{1}{2}$
  - d**  $f(x) = 3x^3 - 5x + 2$  where the tangent has gradient 4
  - e**  $f(x) = ax^2 + bx + c$  where the tangent is horizontal.
- 14** Find the coordinates of point P.

**Example 13****Self Tutor**

The tangent to  $f(x) = 2x^2 - ax + b$  at the point  $(2, 7)$  has a gradient of 3. Find  $a$  and  $b$ .

$$f(x) = 2x^2 - ax + b$$

$$\therefore f'(x) = 4x - a$$

$$\text{Now } f'(2) = 3, \text{ so } 4(2) - a = 3$$

$$\therefore a = 5$$

$$\text{Also, } f(2) = 7, \text{ so } 2(2)^2 - 5(2) + b = 7$$

$$\therefore b = 9$$

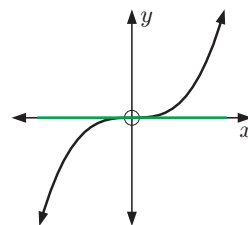
## STATIONARY POINTS OF INFLECTION

It is not always true that whenever we find a value of  $x$  where  $f'(x) = 0$ , we have a local maximum or minimum.

For example,  $f(x) = x^3$  has  $f'(x) = 3x^2$ , so  $f'(x) = 0$  when  $x = 0$ .

The tangent to the curve crosses over the curve at  $O(0, 0)$ . This tangent is horizontal, but  $O(0, 0)$  is neither a local maximum nor a local minimum.

Rather, this point is called a **stationary inflection** (or **inflexion**) as the curve changes its curvature or shape.



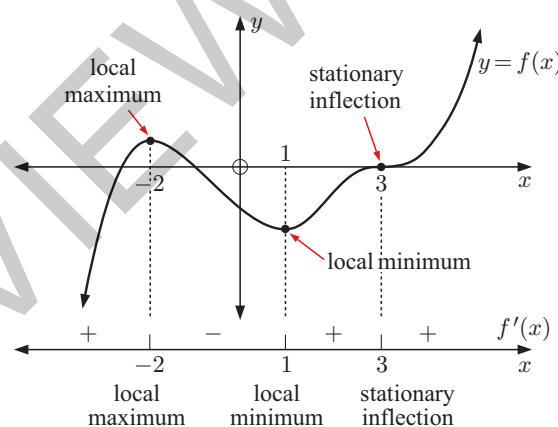
## SIGN DIAGRAMS

In calculus we commonly use sign diagrams of the derivative function  $f'(x)$  to determine the nature of a stationary point.

Consider the graph alongside.

The sign diagram of its gradient function is shown directly beneath it.

The signs on the sign diagram of  $f'(x)$  indicate whether the gradient of  $y = f(x)$  is positive or negative on that interval.



We observe the following properties:

| Stationary point where $f'(a) = 0$ | Sign diagram of $f'(x)$ near $x = a$  | Shape of curve near $x = a$ |
|------------------------------------|---|-----------------------------|
| local maximum                      | $\begin{array}{c} + \quad - \\ \leftarrow \quad \rightarrow \\ a \quad x \end{array} \quad f'(x)$   |                             |
| local minimum                      | $\begin{array}{c} - \quad + \\ \leftarrow \quad \rightarrow \\ a \quad x \end{array} \quad f'(x)$   |                             |
| stationary inflection              | $\begin{array}{c} + \quad + \\ \leftarrow \quad \rightarrow \\ a \quad x \end{array} \quad f'(x) \quad \begin{array}{c} - \quad - \\ \leftarrow \quad \rightarrow \\ a \quad x \end{array} \quad f'(x)$ |                             |

To find the stationary points of a function  $f(x)$ , we find the values of  $x$  for which  $f'(x) = 0$ . The sign diagram of  $f'(x)$  tells us whether each stationary point is a local maximum, local minimum, or stationary inflection.

**Example 1**

As a hot air balloon is inflated, the volume of air inside it after  $t$  minutes is given by  $V = 2t^3 - 3t^2 + 10t + 2 \text{ m}^3$  where  $0 \leq t \leq 8$ .

Find:

- a the initial volume of air in the balloon
- b the volume at  $t = 8$  minutes
- c  $\frac{dV}{dt}$  and explain what it means
- d the rate of increase in volume at  $t = 4$  minutes.



- a When  $t = 0$ ,  $V = 2 \text{ m}^3$   
Initially there were  $2 \text{ m}^3$  of air in the balloon.

- b When  $t = 8$ ,  $V = 2(8)^3 - 3(8)^2 + 10(8) + 2$   
 $= 914 \text{ m}^3$

After 8 minutes there were  $914 \text{ m}^3$  of air in the balloon.

- c  $\frac{dV}{dt} = 6t^2 - 6t + 10 \text{ m}^3 \text{ min}^{-1}$

This function tells us the rate at which the volume of air in the balloon is increasing after  $t$  minutes.

- d When  $t = 4$ ,  $\frac{dV}{dt} = 6(4)^2 - 6(4) + 10$   
 $= 82 \text{ m}^3 \text{ min}^{-1}$

Since  $\frac{dV}{dt} > 0$ ,  $V$  is increasing.

The volume of air in the balloon is increasing at  $82 \text{ m}^3 \text{ min}^{-1}$  at  $t = 4$  minutes.

**EXERCISE 12A.1**

1 Find:

a  $\frac{dM}{dt}$  if  $M = t^3 - 3t^2 + 1$

b  $\frac{dR}{dt}$  if  $R = (2t + 1)^2$

2 a An area  $A$  is measured in  $\text{cm}^2$ , and time  $t$  is measured in seconds.

i What are the units for  $\frac{dA}{dt}$ ?

ii What does  $\frac{dA}{dt}$  mean?

b A volume  $V$  is measured in  $\text{m}^3$ , and time  $t$  is measured in minutes.

i What are the units for  $\frac{dV}{dt}$ ?

ii What does  $\frac{dV}{dt}$  mean?

3 The estimated future profits of a small business are given by  $P(t) = 2t^2 - 12t + 118$  thousand dollars, where  $t$  is the time in years from now.

a What is the current annual profit?

b Find  $\frac{dP}{dt}$  and state its units.

c Find  $\frac{dP}{dt}$  when  $t = 8$ . Explain what this value means.

**Example 8****Self Tutor**

Find:

**a**  $\int_1^2 (x^3 - 4x + 5) dx$

**b**  $\int_1^4 \left(2x + \frac{3}{x^2}\right) dx$

**a** 
$$\begin{aligned} \int_1^2 (x^3 - 4x + 5) dx &= \left[ \frac{1}{4}x^4 - 2x^2 + 5x \right]_1^2 \\ &= \left( \frac{2^4}{4} - 2(2)^2 + 5(2) \right) - \left( \frac{1^4}{4} - 2(1)^2 + 5(1) \right) \\ &= 6 - 3\frac{1}{4} \\ &= \frac{11}{4} \end{aligned}$$

We omit  $c$  for definite integrals.



**b** 
$$\begin{aligned} \int_1^4 \left(2x + \frac{3}{x^2}\right) dx &= \int_1^4 (2x + 3x^{-2}) dx \\ &= \left[ x^2 - \frac{3}{x} \right]_1^4 \\ &= \left( 16 - \frac{3}{4} \right) - (1 - 3) \\ &= 17\frac{1}{4} \end{aligned}$$

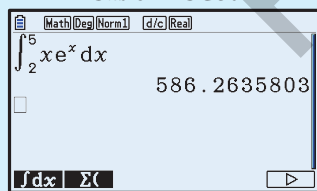
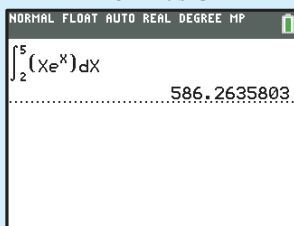
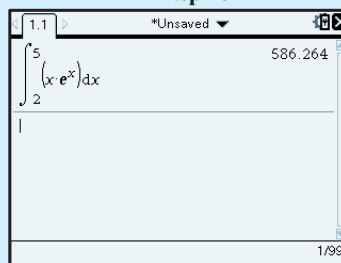
Some definite integrals are difficult or even impossible to evaluate analytically. In these cases you can use a graphics calculator to evaluate the integral.



GRAPHICS  
CALCULATOR  
INSTRUCTIONS

**Example 9****Self Tutor**

Evaluate  $\int_2^5 xe^x dx$  correct to 4 significant figures.

**Casio fx-CG50****TI-84 Plus CE****TI-nspire**

$$\int_2^5 xe^x dx \approx 586.3$$

**EXERCISE 13G****1** Find:

**a**  $\int_1^2 x^3 dx$  and  $\int_1^2 (-x^3) dx$

**b**  $\int_0^1 x^7 dx$  and  $\int_0^1 (-x^7) dx$

## F

# THE MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

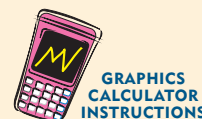
In the **Core Topics SL** book we saw that:

- the **mean** is a measure of the **centre** of a distribution
- the **standard deviation** is a measure of the **spread** of the distribution.

## INVESTIGATION 2

## THE MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

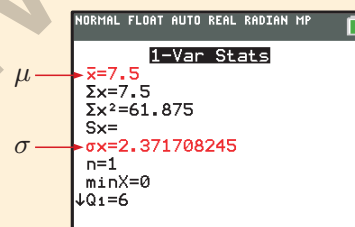
In this Investigation we will use a calculator to calculate the mean and standard deviation of several binomial distributions. A spreadsheet can also be used to speed up the process.



### What to do:

- 1 We will first calculate the mean and standard deviation for the variable  $X \sim B(30, 0.25)$ .

- a Enter the possible values for  $X$  from  $x = 0$  to  $x = 30$  into **List 1**, and their corresponding binomial probabilities  $P(X = x) = \binom{30}{x}(0.25)^x(0.75)^{30-x}$  into **List 2**.
- b Calculate the descriptive statistics for the distribution. You should obtain the results in the screenshot.



- 2 Copy and complete the following table for distributions with other values of  $n$  and  $p$ .

|          | $p = 0.1$ | $p = 0.25$                             | $p = 0.5$ | $p = 0.7$ |
|----------|-----------|--|-----------|-----------|
| $n = 10$ |           |  |           |           |
| $n = 30$ |           | $\mu = 7.5$<br>$\sigma \approx 2.3717$ |           |           |
| $n = 50$ |           |  |           |           |

- 3 Compare your values with the formulae  $\mu = np$  and  $\sigma = \sqrt{np(1-p)}$ .

From the **Investigation** you should have observed the following results:

Suppose  $X$  is a binomial random variable with parameters  $n$  and  $p$ , so  $X \sim B(n, p)$ .

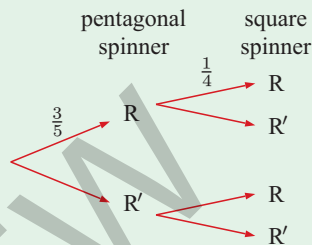
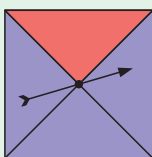
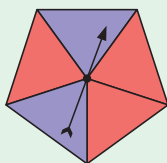
- The **mean** of  $X$  is  $\mu = np$ .
- The **variance** of  $X$  is  $\sigma^2 = np(1-p)$ .
- The **standard deviation** of  $X$  is  $\sigma = \sqrt{np(1-p)}$ .

- 9 Suppose  $X$  is the number of marsupials entering a park at night. It is suspected that  $X$  has a probability mass function  $P(X) = a(x^2 - 8x)$  where  $x = 0, 1, 2, 3, \dots, 8$ .

- Find the constant  $a$ .
- Find the expected number of marsupials entering the park on a given night.



- 10 Consider the two spinners illustrated:



- Copy and complete the tree diagram which shows all possible results when the two spinners are spun together.
  - Calculate the probability that exactly one red will occur.
  - The pair of spinners is now spun 10 times. Let  $X$  be the number of times that exactly one red occurs.
    - State the distribution of  $X$ .
    - Write down expressions for  $P(X = 1)$  and  $P(X = 9)$ . Hence determine which of these outcomes is more likely.
- 11 A school volleyball team has 9 players, each of whom has a 75% chance of coming to any given game. The team needs at least 6 players to avoid forfeiting the game.
- Find the probability that for a randomly chosen game, the team will:
    - have all of its players
    - have to forfeit the game.
  - The team plays 30 games for the season. How many games would you expect the team to forfeit?
- 12 It is observed that 3% of all batteries produced by a company are defective. For a random sample of 20 batteries, calculate the probability that:
- none are defective
  - at least one is defective.

## REVIEW SET 14B

- 1 Sally's number of hits in each softball match has the probability distribution shown.

| $x$        | 0    | 1    | 2   | 3    | 4    | 5    |
|------------|------|------|-----|------|------|------|
| $P(X = x)$ | 0.07 | 0.14 | $k$ | 0.46 | 0.08 | 0.02 |

- State clearly what the random variable represents.
  - Find:
    - $k$
    - $P(X \geq 2)$
    - $P(1 \leq X \leq 3)$
  - Find the mode and median number of hits.
- 2 Show that the following are valid probability mass functions:

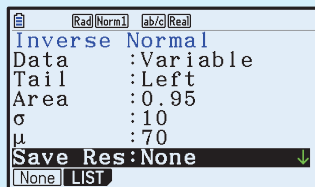
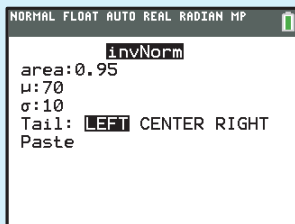
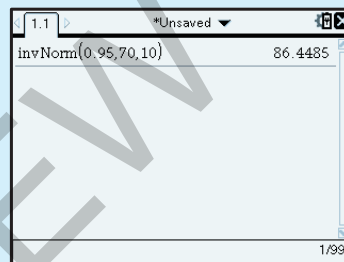
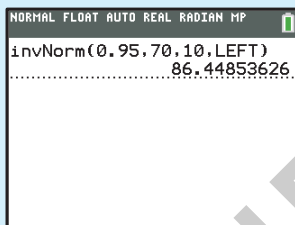
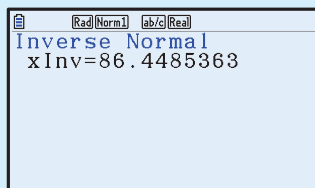
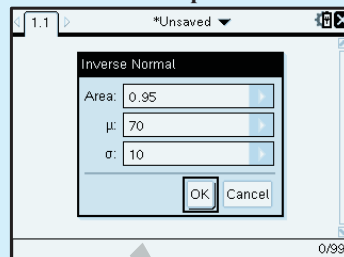
a  $P(x) = \frac{e^x}{1+e}$ ,  $x = 0, 1$

b  $P(x) = \frac{x^2 + x}{40}$ ,  $x = 1, 2, 3, 4$



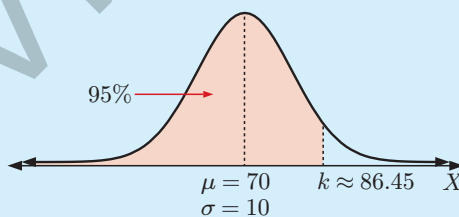
**Example 5****Self Tutor**

A population of crabs has shell length  $X$  mm.  $X$  is normally distributed with mean 70 and standard deviation 10. Find  $k$  for which  $P(X < k) = 0.95$ .

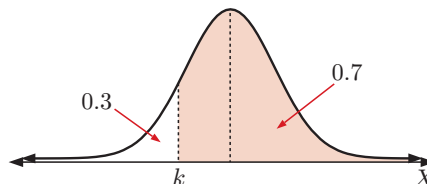
**Casio fx-CG50****TI-84 Plus CE****TI-nspire**

If  $P(X < k) = 0.95$  then  
 $k \approx 86.45$

The 95% quantile corresponds to a shell width of 86.45 mm.



When using the **HP Prime**, **TI-84 Plus CE**, or **TI-nspire** calculators, we must always use the area to the *left* of  $k$ . Therefore, to find  $k$  such that  $P(X > k) = 0.7$ , we instead find  $k$  such that  $P(X < k) = 1 - 0.7 = 0.3$ .

**EXERCISE 15C**

- 1 Suppose  $X$  is normally distributed with mean 20 and standard deviation 3. Illustrate with a sketch and find  $k$  such that:
  - a  $P(X \leq k) = 0.3$
  - b  $P(X \leq k) = 0.9$
  - c  $P(X \leq k) = 0.5$
  - d  $P(X > k) = 0.2$
  - e  $P(X < k) = 0.62$
  - f  $P(X \geq k) = 0.13$
- 2 Given that  $Z \sim N(0, 1^2)$ , illustrate with a sketch and find  $k$  such that:
  - a  $P(Z \leq k) = 0.81$
  - b  $P(Z \leq k) = 0.58$
  - c  $P(Z \leq k) = 0.17$
  - d  $P(Z \geq k) = 0.95$
  - e  $P(Z \geq k) = 0.9$
  - f  $P(Z \geq k) = 0.41$

In our insect repellent example, suppose the researcher was concerned about any change in the average protection time. The researcher decides to test the hypothesis  $H_0: \mu = 6$  against  $H_1: \mu \neq 6$  at a significance level  $\alpha = 0.01$ .

Using the test statistic  $t \approx 3.39$  and  $T \sim t_{50-1}$ ,

the  $p$ -value  $\approx 2 \times P(T \geq |3.39|)$

$$\approx 0.00137$$

Since the  $p$ -value is less than the significance level  $\alpha$ , the researcher has enough evidence to reject  $H_0$ .

The smaller the  $p$ -value, the more evidence there is against  $H_0$ .



### Important:

We **must** choose a significance level **before** we test the hypothesis. Otherwise, we can be tempted to choose a significance level to give the test outcome that we desire. For example, it is *not appropriate* to calculate a  $p$ -value and then select a value of  $\alpha$  so that  $H_0$  will be rejected.

## SUMMARY OF STEPS FOR STUDENT'S $t$ -TEST FOR A POPULATION MEAN

**Step 1:** State the **null hypothesis**  $H_0: \mu = \mu_0$  and **alternative hypothesis**  $H_1$ .

**Step 2:** State the **significance level**  $\alpha$ .

**Step 3:** Calculate the value of the **test statistic**  $t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$ .

**Step 4:** Calculate the  **$p$ -value** using  $T \sim t_{n-1}$  as follows:

- If  $H_1: \mu > \mu_0$ ,  $p\text{-value} = P(T \geq t)$ .
- If  $H_1: \mu < \mu_0$ ,  $p\text{-value} = P(T \leq t)$ .
- If  $H_1: \mu \neq \mu_0$ ,  $p\text{-value} = 2 \times P(T \geq |t|)$ .

**Step 5:** Reject  $H_0$  if  $p\text{-value} < \alpha$ .

**Step 6:** Interpret your decision in the context of the problem. Write your conclusion in a sentence.

### Example 1

### Self Tutor

The manager of a restaurant chain goes to a seafood wholesaler and inspects a large catch of over 50 000 prawns. It is known that the population is normally distributed. He will buy the catch if the mean weight exceeds 55 grams per prawn.

A random sample of 60 prawns is taken. The sample mean weight is 56.2 grams with standard deviation 4.2 grams.

Conduct a one-tailed hypothesis test with significance level  $\alpha = 0.05$  to determine whether the manager should purchase the catch.

**Step 1:** Let  $\mu$  be the population mean weight per prawn.

The hypotheses that should be considered are:

$H_0: \mu = 55$  {the mean weight is 55 grams per prawn}

$H_1: \mu > 55$  {the mean weight exceeds 55 grams per prawn}

**Step 2:** The significance level is  $\alpha = 0.05$ .

The critical value  $\chi^2_{\text{crit}}$  depends on the degrees of freedom (df) and the significance level  $\alpha$  of the test.

The table alongside shows the critical value for various values of df and  $\alpha$ .

Click on the icon for a more detailed table of critical values.



| Degrees of freedom (df) | Significance level $\alpha$ |       |       |
|-------------------------|-----------------------------|-------|-------|
|                         | 10%                         | 5%    | 1%    |
| 1                       | 2.71                        | 3.84  | 6.63  |
| 2                       | 4.61                        | 5.99  | 9.21  |
| 3                       | 6.25                        | 7.81  | 11.34 |
| 4                       | 7.78                        | 9.49  | 13.28 |
| 5                       | 9.24                        | 11.07 | 15.09 |
| 6                       | 10.64                       | 12.59 | 16.81 |
| 7                       | 12.02                       | 14.07 | 18.48 |
| 8                       | 13.36                       | 15.51 | 20.09 |
| 9                       | 14.68                       | 16.92 | 21.67 |
| 10                      | 15.99                       | 18.31 | 23.21 |

## $\chi^2$ GOODNESS OF FIT TEST PROCEDURE WITH CRITICAL VALUES

**Step 1:** State the **null hypothesis**  $H_0$  and the **alternative hypothesis**  $H_1$ .

**Step 2:** State the **significance level**  $\alpha$ .

**Step 3:** Calculate the value of the **test statistic**:  $\chi^2_{\text{calc}} = \sum \frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$ .

**Step 4:** Find the **critical value**  $\chi^2_{\text{crit}}$  where df = number of categories - 1.

**Step 5:** Reject  $H_0$  if  $\chi^2_{\text{calc}} \geq \chi^2_{\text{crit}}$ .

**Step 6:** Interpret your decision in the context of the problem. Write your conclusion in a sentence.

### EXERCISE 16D.2

- 1** A goodness of fit test is to be used to test the following hypotheses:

$$H_0: p_1 = \frac{1}{5}, p_2 = \frac{1}{5}, \dots, p_5 = \frac{1}{5}$$

$$H_1: \text{at least one of } p_1, p_2, \dots, p_5 \neq \frac{1}{5} \text{ with a 5\% level of significance.}$$

The value of the test statistic is  $\chi^2_{\text{calc}} = 10.3$ .

- Write down the critical value  $\chi^2_{\text{crit}}$  for the test.
- Hence determine whether  $H_0$  should be rejected.

- 2** Lucy bought a large packet of Chewy Chews lollies. She counted the number of lollies of each colour it contained. Her results are shown in the table.

The manufacturer of Chewy Chews claims that they make equal quantities of each colour.

Lucy wants to test the manufacturer's claim with a goodness of fit test using a 10% level of significance.

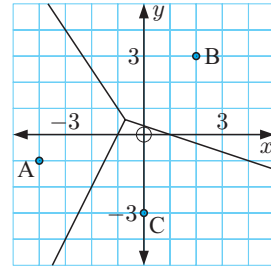
- Write down the hypotheses that Lucy should test.
- Calculate  $\chi^2_{\text{calc}}$ .
- Find the critical value  $\chi^2_{\text{crit}}$  for the test.
- Does Lucy have enough evidence to reject the manufacturer's claim?
- Verify your answer by examining the  $p$ -value.

| Colour | Frequency |
|--------|-----------|
| red    | 12        |
| yellow | 17        |
| green  | 20        |
| blue   | 16        |
| Total  | 65        |

## EXERCISE 17A

- 1 Consider this Voronoi diagram for the sites A, B, and C.

- a State the number of:
- i cells
  - ii edges
  - iii vertices.
- b The point P has coordinates  $(-1, 2)$ .
- i Which site is closest to P? Explain your answer.
  - ii Verify your answer by calculating the distance between P and each site.

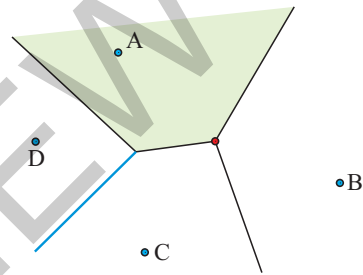


- c The point Q has coordinates  $(-1, 0)$  and lies on an edge of the Voronoi diagram.
- i Identify the site(s) closest to Q. Explain your answer.
  - ii Verify your answer by calculating the distance between Q and each site.

- 2 This Voronoi diagram has been constructed for the sites A, B, C, and D.

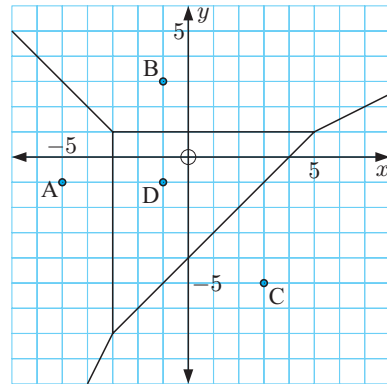
What can you say about:

- a the points which lie in the green cell
- b the points which lie on the blue edge
- c the red vertex?



- 3 Consider this Voronoi diagram for the sites A, B, C, and D.

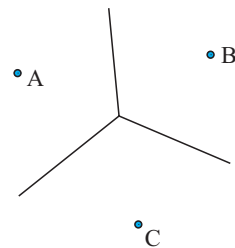
- a Identify the site which is closest to:
- i  $(2, 3)$
  - ii  $(-1, -4)$
  - iii  $(6, 0)$
  - iv  $(-4, -3)$
- b Verify that:
- i  $(-3, 0)$  is equidistant from A and D
  - ii  $(-3, 2)$  is also equidistant from A and D.
- c Explain why  $(-3, 0)$  lies on an edge of the Voronoi diagram, but  $(-3, 2)$  does not.
- d Find the area of cell D.



- 4 In the Voronoi diagram alongside, suppose point  $P_A$  lies in cell A, and  $P_B$  lies in cell B.

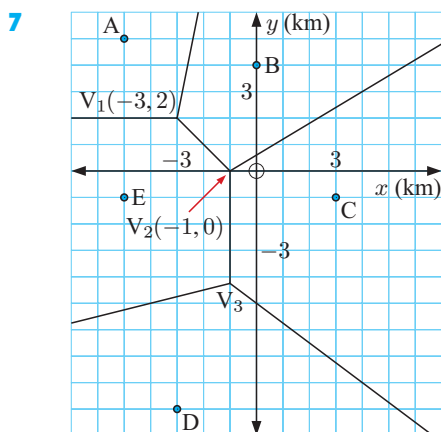
Discuss whether the following statements are necessarily true:

- a  $P_A$  is closer to A than to any other site.
- b  $P_B$  is closer to B than to C.
- c A is closer to  $P_A$  than to  $P_B$ .
- d B is closer to  $P_B$  than to C.



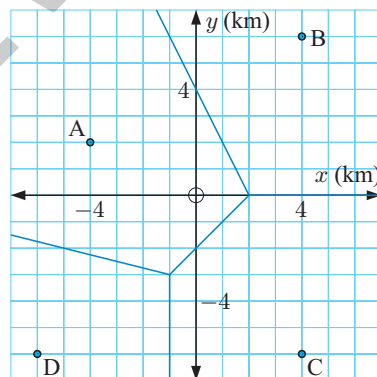
- 5 Let X be a site on a Voronoi diagram, and P be a point within the interior of cell X. Suppose a circle is drawn with centre P, passing through X. Explain why this circle cannot contain any other sites.

- 6 Consider the sites  $A(-2, 7)$ ,  $B(4, 7)$ ,  $C(6, 3)$ , and  $D(-2, -5)$ .
- Construct a Voronoi diagram for these sites.
  - Use your diagram to find the coordinates of the vertices.
  - Find the centre and radius of the largest empty circle.



This Voronoi diagram shows the police stations in a city.

- Which station is closest to:
    - $(3, 4)$
    - $(-3, -3)$ ?
  - Find the coordinates of the vertex  $V_3$ .
  - A new police station is to be built. To maximise the efficiency of the stations, the new station will be built as far as possible from the existing stations.
    - Where should the new station be built?
    - Show that this new station will now be the closest station to  $(-3, -3)$ .
- 8 Brigitte would like to open a burger store. She has drawn a Voronoi diagram showing the existing burger stores in her area so she can locate her own store as far from the existing stores as possible.
- Find the optimal position for Brigitte's store, and state how far this location is from her nearest competitor.
  - Just before Brigitte purchases the site, a new burger store opens at  $(4, 2)$ .
    - Redraw the Voronoi diagram with the new store added.
    - Find the new optimal position for Brigitte's store.
- 9 Answer the **Opening Problem** on page 418.

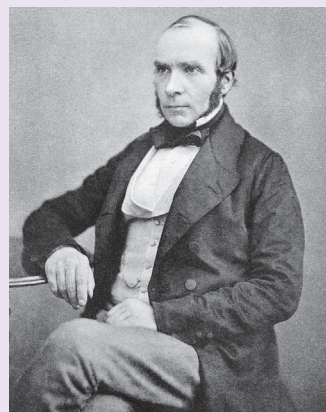


## HISTORICAL NOTE

## THE BROAD STREET CHOLERA OUTBREAK

In 1854 in the Soho district of London, there was a serious cholera outbreak which killed 616 people. At the time, the cause of cholera was not fully understood. It was thought under the *miasma theory* that diseases such as cholera or the Black Death were caused by some kind of "bad air". The germ theory was not proposed until 1861 by **Louis Pasteur**.

In this outbreak, the English physician **John Snow** looked at the pattern of illness, and decided that the disease was spread by something in the water. At that time, water was drawn directly from the Thames River which was highly contaminated. Snow identified the source of the outbreak as the public water pump on Broad Street, and he persuaded the authorities to disable the pump by removing its handle.



John Snow