HAESE MATHEMATICS

Specialists in mathematics education

Mathematics

Analysis and Approaches HL 2

Michael Haese
Mark Humphries
Chris Sangwin
Ngoc Vo

for use with
IB Diploma
Programme

ACTIVITY 3

A **trigonometric series** is the sum of a sequence of trigonometric expressions which follow a rule.

In this Investigation we will explore patterns formed by a trigonometric series.

What to do:

1 Consider the function $f(x)=\sin x+\dfrac{\sin 3x}{3}$.

a Show that $f(x)=\frac{2}{3}\sin x(2\cos^2 x+1)$.

Hint: Write $\sin 3x$ as $\sin(2x+x)$.

b Hence find the x-intercepts of $y=f(x)$ on $-4\pi\leqslant x\leqslant 4\pi$.

c Use the graphing package to sketch $y=f(x)$ on $-4\pi\leqslant x\leqslant 4\pi$. Discuss the similarities and differences between this graph and the graph of $y=\sin x$.

GRAPHING PACKAGE

2 **a** Write the function $f(x)=\sin x+\dfrac{\sin 3x}{3}+\dfrac{\sin 5x}{5}$ in terms of $\sin x$ and $\cos x$.

b Hence find the x-intercepts of this function on $-4\pi\leqslant x\leqslant 4\pi$.

c Use the graphing package to sketch the graph of the function on $-4\pi\leqslant x\leqslant 4\pi$. Compare your graph with the graphs in **1**.

3 Use the graphing package to graph on $-4\pi\leqslant x\leqslant 4\pi$:

a $f(x)=\sin x+\dfrac{\sin 3x}{3}+\dfrac{\sin 5x}{5}+\dfrac{\sin 7x}{7}$

b $f(x)=\sin x+\dfrac{\sin 3x}{3}+\dfrac{\sin 5x}{5}+\dfrac{\sin 7x}{7}+\dfrac{\sin 9x}{9}+\dfrac{\sin 11x}{11}$

4 Predict the graph of $f(x)=\displaystyle\sum_{k=1}^{\infty}\frac{\sin(2k-1)}{2k-1}$.

REVIEW SET 1A

1 Without using a calculator, find:

a $\operatorname{cosec}\frac{\pi}{4}$ **b** $\cot\frac{5\pi}{6}$ **c** $\sec\frac{5\pi}{3}$

2 Find, giving your answer in radians:

a $\arccos\frac{1}{\sqrt{2}}$ **b** $\arctan\frac{1}{\sqrt{3}}$ **c** $\arcsin\left(-\frac{1}{2}\right)$

3 Solve for $0\leqslant x\leqslant 2\pi$:

a $\sec x=\sqrt{2}$ **b** $\sqrt{3}\cos x\operatorname{cosec} x+1=0$

4 Simplify:

a $3\cos(-\theta)-2\cos\theta$ **b** $\cos\left(\frac{3\pi}{2}-\theta\right)$ **c** $\sin\left(\theta+\frac{\pi}{2}\right)$

d $\dfrac{1-\cos^2\theta}{1+\cos\theta}$ **e** $\dfrac{\sin\alpha-\cos\alpha}{\sin^2\alpha-\cos^2\alpha}$ **f** $\dfrac{4\sin^2\alpha-4}{8\cos\alpha}$

5 If $\sin\alpha=-\frac{3}{4}$, $\pi\leqslant\alpha\leqslant\frac{3\pi}{2}$, find the exact value of:

a $\cos\alpha$ **b** $\sin 2\alpha$ **c** $\cos 2\alpha$ **d** $\tan 2\alpha$ **e** $\cos\frac{\alpha}{2}$ **f** $\sin\frac{\alpha}{2}$

E PROPERTIES OF COMPLEX CONJUGATES

INVESTIGATION PROPERTIES OF CONJUGATES

The purpose of this Investigation is to discover any properties that complex conjugates might have.

What to do:

1 Given $z_1 = 1 - i$ and $z_2 = 2 + i$ find:

a z_1^* **b** z_2^* **c** $(z_1^*)^*$ **d** $(z_2^*)^*$

e $z_1 + z_1^*$ **f** $z_2 + z_2^*$ **g** $z_1 z_1^*$ **h** $z_2 z_2^*$

i $(z_1 + z_2)^*$ **j** $z_1^* + z_2^*$ **k** $(z_1 - z_2)^*$ **l** $z_1^* - z_2^*$

m $(z_1 z_2)^*$ **n** $z_1^* z_2^*$ **o** $\left(\dfrac{z_1}{z_2}\right)^*$ **p** $\dfrac{z_1^*}{z_2^*}$

q $(z_1^2)^*$ **r** $(z_1^*)^2$ **s** $(z_2^3)^*$ **t** $(z_2^*)^3$

2 Repeat **1** with z_1 and z_2 of your choice.

3 Examine your results from **1** and **2**, and hence suggest some rules for complex conjugates.

From the **Investigation** you should have discovered the following rules for complex conjugates:

- $(z^*)^* = z$
- $(z_1 + z_2)^* = z_1^* + z_2^*$ and $(z_1 - z_2)^* = z_1^* - z_2^*$
- $(z_1 z_2)^* = z_1^* \times z_2^*$ and $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}, \quad z_2 \neq 0$
- $(z^n)^* = (z^*)^n$ for positive integers n
- $z + z^*$ and zz^* are real.

Example 8 Self Tutor

Show that for all complex numbers z_1 and z_2:

a $(z_1 + z_2)^* = z_1^* + z_2^*$

b $(z_1 z_2)^* = z_1^* \times z_2^*$

a Let $z_1 = a + bi$ and $z_2 = c + di$

$\therefore z_1^* = a - bi$ and $z_2^* = c - di$

Now $z_1 + z_2 = (a + c) + (b + d)i$

$$\begin{aligned} \therefore (z_1 + z_2)^* &= (a + c) - (b + d)i \\ &= a + c - bi - di \\ &= a - bi + c - di \\ &= z_1^* + z_2^* \end{aligned}$$

b Let $z_1 = a + bi$ and $z_2 = c + di$

$$\begin{aligned} \therefore z_1 z_2 &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

$\therefore (z_1 z_2)^* = (ac - bd) - i(ad + bc)$ .... (1)

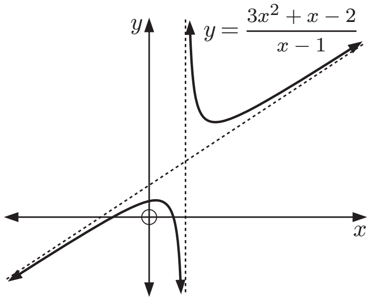
Also, $z_1^* \times z_2^*$

$$\begin{aligned} &= (a - bi)(c - di) \\ &= ac - adi - bci + bdi^2 \\ &= (ac - bd) - i(ad + bc) \quad .... (2) \end{aligned}$$

From (1) and (2), $(z_1 z_2)^* = z_1^* \times z_2^*$

EXERCISE 6D.2

1 The graph of $y = \dfrac{3x^2+x-2}{x-1}$ is shown alongside.



a Find the equation of the vertical asymptote.

b Find the axes intercepts.

c Write the function in the form $y = px + q + \dfrac{r}{x-1}$, and hence find the equation of the oblique asymptote.

2 For each of the following functions:

i Find the equation of the vertical asymptote.

ii Find the axes intercepts.

iii Find the oblique asymptote.

iv Draw a sign diagram of the function.

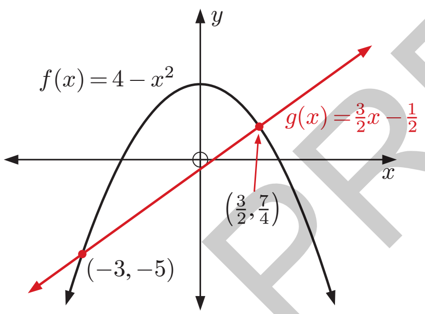
v Hence discuss the behaviour of the function near its asymptotes.

vi Sketch the graph of the function.

a $y = \dfrac{x^2+x-6}{x-1}$ **b** $f(x) = \dfrac{-x^2+3x-2}{x}$ **c** $y = \dfrac{2x^2-8x+8}{x-3}$

d $f(x) = \dfrac{-6x^2+5x}{2x+1}$ **e** $y = \dfrac{-6x^2-4x-1}{3x+2}$ **f** $f(x) = \dfrac{8x^2-19x-15}{1-2x}$

3



Consider the graphs of $y = f(x)$ and $y = g(x)$ shown alongside.

a For $y = \dfrac{f(x)}{g(x)}$ find the equation of the:

i vertical asymptote **ii** oblique asymptote.

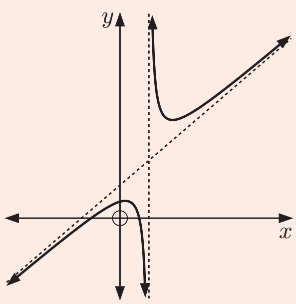
b Copy the graph, and on the same set of axes, graph $y = \dfrac{f(x)}{g(x)}$. Indicate clearly where the x-intercept(s) and asymptote(s) occur.

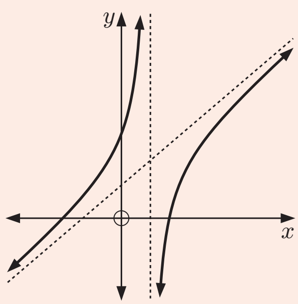
4 Find the equation of the oblique asymptote of $y = \dfrac{ax^2+bx+c}{dx+e}$.

DISCUSSION

You should have noticed that the graph of $y = \dfrac{ax^2+bx+c}{dx+e}$ takes one of the two shapes shown.

What determines which shape the graph takes?





Example 7 **Self Tutor**

Consider the binomial expansion of $\dfrac{1}{\sqrt{2+x}}$.

a Write down the first 4 terms of the expansion.

b State the interval of convergence for the complete expansion.

c Use the expansion to estimate $\dfrac{1}{\sqrt{2.1}}$. Check your answer by direct calculation.

a
$$\begin{aligned}\frac{1}{\sqrt{2+x}} &= (2+x)^{-\frac{1}{2}} \\ &= 2^{-\frac{1}{2}}\left(1+\frac{x}{2}\right)^{-\frac{1}{2}} \\ &= \tfrac{1}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-\frac{1}{2}} \\ &= \tfrac{1}{\sqrt{2}}\sum_{r=0}^{\infty}\binom{-\frac{1}{2}}{r}\left(\frac{x}{2}\right)^r \\ &= \tfrac{1}{\sqrt{2}}\left(1+\left(-\tfrac{1}{2}\right)\left(\frac{x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3+\dots\right) \\ &= \tfrac{1}{\sqrt{2}}\left(1-\tfrac{1}{4}x+\tfrac{3}{32}x^2-\tfrac{5}{128}x^3+\dots\right)\end{aligned}$$

b The series converges provided $\left|\dfrac{x}{2}\right|<1$, which is the interval $-2<x<2$.

c Letting $x=0.1$,
$$\begin{aligned}\frac{1}{\sqrt{2.1}} &\approx \tfrac{1}{\sqrt{2}}\left(1-\tfrac{1}{4}(0.1)+\tfrac{3}{32}(0.1)^2-\tfrac{5}{128}(0.1)^3\right) \\ &\approx 0.690\,064\end{aligned}$$

Using technology, $\dfrac{1}{\sqrt{2.1}}\approx 0.690\,066$.

EXERCISE 8C

1 Evaluate:

a $\binom{\frac{1}{2}}{1}$ **b** $\binom{-\frac{1}{2}}{2}$ **c** $\binom{-2}{2}$ **d** $\binom{-\frac{1}{3}}{0}$ **e** $\binom{-1}{3}$ **f** $\binom{\frac{1}{3}}{3}$

2 Consider the binomial expansion of $\dfrac{1}{1+x}$.

a Write down the first 4 terms.

b State the interval of convergence for the complete expansion.

c Use the expansion to estimate $\dfrac{1}{1.1}$. Check your answer by direct calculation.

3 Consider the binomial expansion of $\dfrac{1}{\sqrt{4-x}}$.

a Write down the first 4 terms.

b State the interval of convergence for the complete expansion.

c Use the expansion to estimate $\dfrac{1}{\sqrt{3.6}}$. Check your answer by direct calculation.

7 **Dirichlet's theorem on arithmetic progressions** states that for any two positive integers a and d which have no common factors, there are infinitely many prime numbers of the form $a+nd$ where $n \in \mathbb{Z}^+$.
Prove Dirichlet's theorem on arithmetic progressions for the case where $a=3$ and $d=4$.

DISCUSSION

1. Can a proof by contradiction be considered a proof by deduction or proof by equivalence?
2. Is an indirect proof "cheating"?
3. Why is it so hard to prove that π is irrational?

DISCUSSION

A proof is a useful way to present a solution. However, sometimes it is not clear from the proof how someone came up with the idea. The mathematician **Carl Gauss** supposedly said "the architect does not leave scaffolding behind".

For example, consider this proof that an irrational number raised to an irrational power can be rational.

Proof:

Let $I = \sqrt{2}^{\sqrt{2}}$. I is either rational or irrational.

If I is rational then since $\sqrt{2}$ is irrational, I is itself an example of an irrational number to an irrational power being rational.

If I is irrational, consider $I^{\sqrt{2}} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^2 = 2.$

In this case $I^{\sqrt{2}}$ is an example of an irrational number to an irrational power being rational.

By exhaustion, there must be an irrational number which raised to an irrational power, is rational.

In this proof, we do not actually establish whether I is rational. We do not need to.

List the different forms of proof you have learned. Discuss what features of a problem point you towards using particular forms of proof.

REVIEW SET 9A

1. The roots of $f(x) = x^2 + px + q$ are a and b. Prove that $q = ab$ and $p = -(a+b)$.
2. Prove that:
   a $2.\overline{9} \in \mathbb{Z}$
   b $0.\overline{38} \in \mathbb{Q}$
3. State the negation of each statement:
   a The boy has blue eyes.
   b x is larger than 4.
4. State, with justification, whether each statement is true or false.
   a If a function f is periodic with period p, then $f(x+p) = f(x)$ for all x.
   b If $f(x+p) = f(x)$ for all x, then f is periodic with period p.
   c f is periodic with period p if and only if $f(x+p) = f(x)$ for all x.

13 **a** Prove that $1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \dots + n \times n! = (n+1)! - 1$ for all $n \in \mathbb{Z}^+$, where $n!$ is the product of the first n positive integers.

b Prove that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \dots + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$ for all $n \in \mathbb{Z}^+$.

c Hence find the sum $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \dots + \frac{9}{10!}$ in rational form.

14 Prove that $n + 2(n-1) + 3(n-2) + \dots + (n-2)3 + (n-1)2 + n = \frac{n(n+1)(n+2)}{6}$ for all integers $n \geqslant 1$.

Hint: $1 \times 6 + 2 \times 5 + 3 \times 4 + 4 \times 3 + 5 \times 2 + 6 \times 1$
$= 1 \times 5 + 2 \times 4 + 3 \times 3 + 4 \times 2 + 5 \times 1 + (1 + 2 + 3 + 4 + 5 + 6)$

15 The nth term of the sequence $3, \frac{4}{3}, \frac{11}{18}, \dots$ is given by $u_n = a\left(\frac{1}{3}\right)^n + b\left(\frac{1}{2}\right)^n$.

a Find the values of a and b.

b Use the principle of mathematical induction to prove that the sum of the first n terms of this sequence is, $\sum_{i=1}^{n} u_i = \frac{3}{2}\left(1 - \frac{1}{3^n}\right) + 4\left(1 - \frac{1}{2^n}\right)$ for all $n \in \mathbb{Z}^+$.

c Hence find the sum of the infinite series.

Example 8 **Self Tutor**

A sequence is defined by $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for all $n \in \mathbb{Z}^+$.
Prove that $u_n = 2^n - 1$ for all $n \in \mathbb{Z}^+$.

P_n is: if $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for all $n \in \mathbb{Z}^+$, then $u_n = 2^n - 1$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $u_1 = 2^1 - 1 = 1$
$\therefore$ P_1 is true.

(2) If P_k is true, then $u_k = 2^k - 1$

Now
$$\begin{aligned} u_{k+1} &= 2u_k + 1 \\ &= 2(2^k - 1) + 1 \quad \{\text{using } P_k\} \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

$\therefore$ P_{k+1} is also true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
P_n is true for all $n \in \mathbb{Z}^+$. {principle of mathematical induction}

16 Use the principle of mathematical induction to prove these propositions:

a If a sequence $\{u_n\}$ is defined by $u_1 = 5$ and $u_{n+1} = u_n + 8n + 5$ for all $n \in \mathbb{Z}^+$, then $u_n = 4n^2 + n$.

b If the first term of a sequence is 1, and subsequent terms are defined by the recursion formula $u_{n+1} = 2 + 3u_n$, then $u_n = 2(3^{n-1}) - 1$.

Example 16 **Self Tutor**

If $\mathbf{p} = 3\mathbf{i} - 5\mathbf{j}$ and $\mathbf{q} = -\mathbf{i} - 2\mathbf{j}$, find $|\,\mathbf{p} - 2\mathbf{q}\,|$.

$$\begin{aligned}\mathbf{p} - 2\mathbf{q} &= 3\mathbf{i} - 5\mathbf{j} - 2(-\mathbf{i} - 2\mathbf{j}) \\ &= 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{i} + 4\mathbf{j} \\ &= 5\mathbf{i} - \mathbf{j}\end{aligned}$$

$$\therefore\ |\,\mathbf{p} - 2\mathbf{q}\,| = \sqrt{5^2 + (-1)^2} = \sqrt{26} \text{ units}$$

7 If $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{s} = -\mathbf{i} + 4\mathbf{j}$, find:

a $|\,\mathbf{r}\,|$ **b** $|\,\mathbf{s}\,|$ **c** $|\,\mathbf{r} + \mathbf{s}\,|$ **d** $|\,\mathbf{r} - \mathbf{s}\,|$ **e** $|\,\mathbf{s} - 2\mathbf{r}\,|$

f $|\,2\mathbf{r}\,|$ **g** $|\,\mathbf{r} + 2\mathbf{s}\,|$ **h** $|\,2\mathbf{r} - \mathbf{s}\,|$ **i** $|\,\mathbf{s} - 2\mathbf{r}\,|$

8 **a** If $\mathbf{p} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, find:

i $|\,\mathbf{p}\,|$ **ii** $|\,2\mathbf{p}\,|$ **iii** $|\,-2\mathbf{p}\,|$ **iv** $|\,3\mathbf{p}\,|$ **v** $|\,-3\mathbf{p}\,|$

b By letting $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, prove that $|\,k\mathbf{v}\,| = |\,k\,|\,|\,\mathbf{v}\,|$.

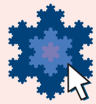
9 Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

Show by equating components, that if $k\mathbf{x} = \mathbf{a}$ then $\mathbf{x} = \frac{1}{k}\mathbf{a}$.

ACTIVITY **SKI RACER**

Click on the icon to practise your skills with vectors.

SKI RACER



F VECTORS IN SPACE

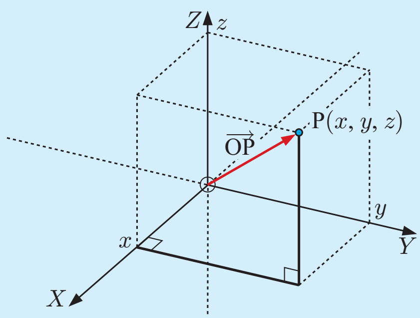
In **Chapter 10** of the **Core Topics** book, we saw that we can specify a point in 3-dimensional space using 3 mutually perpendicular axes called the X, Y, and Z axes.

In 3-dimensional space, the **position vector** of $\mathrm{P}(x, y, z)$

is $\overrightarrow{\mathrm{OP}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

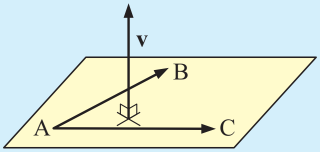
where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

are the **base unit vectors** in the X, Y, and Z directions respectively.



Example 37 Self Tutor

Find a vector which is perpendicular to the plane passing through the points A(1, −1, 2), B(3, 1, 0), and C(−1, 2, −3).



$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}$

The vector $\mathbf{v}$ must be perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\therefore \quad \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -2 \\ -2 & 3 & -5 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 3 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -2 \\ -2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix} \mathbf{k}$$

$$= -4\mathbf{i} + 14\mathbf{j} + 10\mathbf{k}$$

$$= -2(2\mathbf{i} - 7\mathbf{j} - 5\mathbf{k})$$

Any non-zero multiple of $\begin{pmatrix} 2 \\ -7 \\ -5 \end{pmatrix}$ will be perpendicular to the plane.

12 Find a vector which is perpendicular to the plane passing through the points:

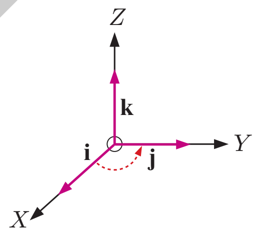
a A(1, 3, 2), B(0, 2, −5), C(3, 1, −4) **b** P(2, 0, −1), Q(0, 1, 3), R(1, −1, 1)

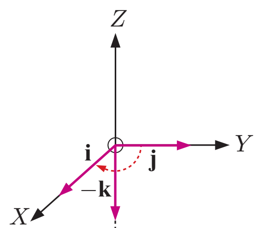
THE DIRECTION OF $\mathbf{a} \times \mathbf{b}$

We have already observed that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$, so $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ are in opposite directions.

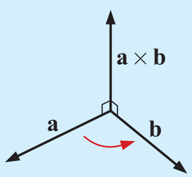
However, what is the direction of each?

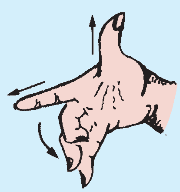
In the last Exercise, we saw that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$.





In general, the **direction** of $\mathbf{a} \times \mathbf{b}$ is determined by the **right hand rule**:





If the fingers on your right hand turn from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

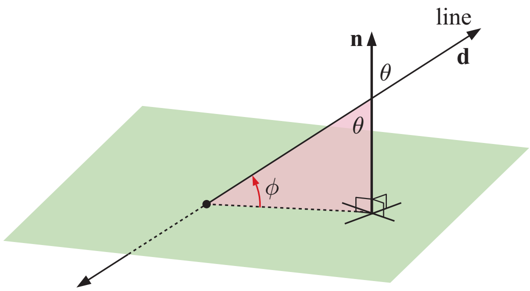
THE LENGTH OF $\mathbf{a} \times \mathbf{b}$

Using $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$, $|\mathbf{a} \times \mathbf{b}| = \sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}$.

H ANGLES IN SPACE

THE ANGLE BETWEEN A LINE AND A PLANE

Suppose a line has direction vector $\mathbf{d}$ and a plane has normal vector $\mathbf{n}$. We allow $\mathbf{n}$ to intersect the line, making an angle of θ with it. The angle between the line and the plane is the acute angle ϕ.



Now $\sin\phi = \cos\theta = \dfrac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}||\mathbf{d}|}$

$$\therefore \quad \phi = \sin^{-1}\left(\frac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}||\mathbf{d}|}\right)$$

Example 19 **Self Tutor**

Find the angle between the plane $x + 2y - z = 8$ and the line with equations $x = t,\ y = 1 - t,\ z = 3 + 2t,\ t \in \mathbb{R}$.

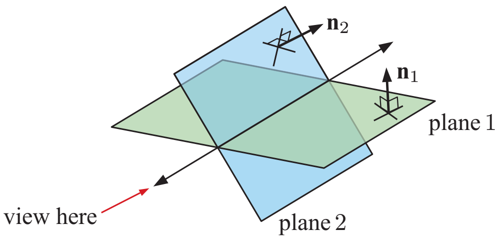
$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

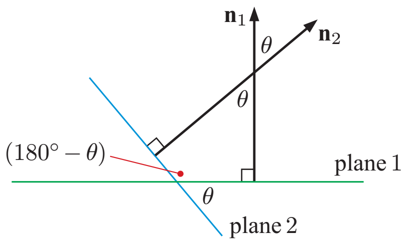
The angle between the plane and the line is $\phi = \sin^{-1}\left(\dfrac{|1-2-2|}{\sqrt{1+4+1}\sqrt{1+1+4}}\right)$

$= \sin^{-1}\left(\dfrac{3}{\sqrt{6}\sqrt{6}}\right)$

$= \sin^{-1}\left(\frac{1}{2}\right) = 30°$

THE ANGLE BETWEEN TWO PLANES





The cosine of the acute angle between the two normals is $\cos\theta = \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$.

If two planes have normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, and θ is the acute angle between them, then $\theta = \cos^{-1}\left(\dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$.

E EULER'S FORM

HISTORICAL NOTE — EULER'S BEAUTIFUL EQUATION

One of the most remarkable results in mathematics is known as **Euler's beautiful equation** $e^{i\pi} = -1$ named after **Leonhard Euler**.

It is called beautiful because it links together three great constants of mathematics: Euler's constant e, the imaginary number i, and the ratio of a circle's circumference to its diameter π.

Harvard lecturer **Benjamin Pierce** said of $e^{i\pi} = -1$,

> "Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth."

Euler's beautiful equation is a special case of a more general result that Euler proved:

For any $\theta \in \mathbb{R}$, $\boldsymbol{e^{i\theta} = \cos\theta + i\sin\theta}$.

This identity allows us to write any complex number $z = |z| \operatorname{cis}\theta$ in the **Euler form** $z = |z|e^{i\theta}$.

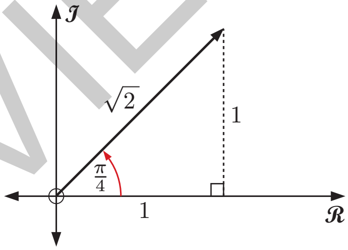
For example, consider $z = 1 + i$.

$|z| = \sqrt{2}$ and $\theta = \frac{\pi}{4}$,

$\therefore \quad 1 + i = \sqrt{2}\operatorname{cis}\frac{\pi}{4} = \sqrt{2}e^{i\frac{\pi}{4}}$

So, $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ is the **polar form** of $1 + i$

and $\sqrt{2}e^{i\frac{\pi}{4}}$ is the **Euler form** of $1 + i$.



We cannot prove the Euler identity for ourselves until we study calculus. However, in the following **Investigation** we put together our knowledge of exponentials and complex numbers to justify why it is reasonable.

INVESTIGATION — EULER'S FORM

In our previous study of exponentials we saw how the exponential e is related to compound interest. In particular:

- For 100% compound growth with interest calculated in n subintervals, the value after each subinterval is generated using the sequence $1, \left(1+\frac{1}{n}\right), \left(1+\frac{1}{n}\right)^2, ..., \left(1+\frac{1}{n}\right)^n$. We saw that as $n \to \infty$, $\left(1+\frac{1}{n}\right)^n \to e^1$.
- If interest is paid continuously, then the compounded amount after 1 year is the initial amount multiplied by e^R, where R is the interest rate.

We now consider the idea of *imaginary* compound growth.

What to do:

1 Consider the sequence $1, \left(1+\frac{i}{n}\right), \left(1+\frac{i}{n}\right)^2, ..., \left(1+\frac{i}{n}\right)^n$ which has $n+1$ terms. This sequence represents imaginary compound growth with interest calculated in n subintervals.

D TRIGONOMETRIC LIMITS

INVESTIGATION EXAMINING $\frac{\sin\theta}{\theta}$ NEAR $\theta = 0$

In this Investigation we examine $\frac{\sin\theta}{\theta}$ when θ is close to 0 and θ is in radians.

What to do:

1 **a** Graph $y = \frac{\sin x}{x}$ for $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ using a graphics calculator or graphing package.

GRAPHING PACKAGE

b Discuss the behaviour of the graph as x approaches 0.

2 We now prove the result $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

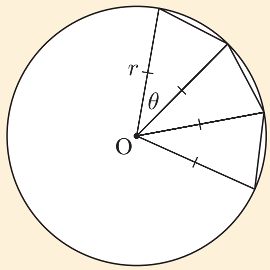
a Show that $f(\theta) = \frac{\sin\theta}{\theta}$ is an even function. What does this mean graphically?

b Suppose $f(\theta)$ is an even function for which $\lim\limits_{\theta \to 0^+} f(\theta) = A$.

c Explain why: **i** $\lim\limits_{\theta \to 0^-} f(\theta) = A$ **ii** $\lim\limits_{\theta \to 0} f(\theta) = A$

d Since $\frac{\sin\theta}{\theta}$ is even, we need only examine $\frac{\sin\theta}{\theta}$ for positive θ.

A circle of radius r contains n congruent isosceles triangles as shown.



i Use the diagram to show that $\lim\limits_{n \to \infty} \frac{n}{2} r^2 \sin\frac{2\pi}{n} = \pi r^2$.

ii Hence show that:

(1) $\lim\limits_{n \to \infty} \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} = 1$ **(2)** $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

From the **Investigation**, you should have found that:

If θ is in radians, then $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Example 6 Self Tutor

Find $\lim\limits_{\theta \to 0} \frac{\sin 3\theta}{\theta}$.

$$\begin{aligned}
\lim_{\theta \to 0} \frac{\sin 3\theta}{\theta} &= \lim_{\theta \to 0} \frac{\sin 3\theta}{3\theta} \times 3 \\
&= 3 \times \lim_{3\theta \to 0} \frac{\sin 3\theta}{3\theta} \qquad \{\text{as } \theta \to 0,\ 3\theta \to 0 \text{ also}\} \\
&= 3 \times 1 \\
&= 3
\end{aligned}$$

Example 19 — Self Tutor

If y is a function of x, find: **a** $\dfrac{d}{dx}\left(\dfrac{1}{y}\right)$ **b** $\dfrac{d}{dx}(xy^2)$

a
$$\frac{d}{dx}\left(\frac{1}{y}\right) = \frac{d}{dx}(y^{-1}) = -y^{-2}\frac{dy}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

b
$$\frac{d}{dx}(xy^2) = 1 \times y^2 + x \times 2y\frac{dy}{dx} \quad \{\text{product rule}\}$$
$$= y^2 + 2xy\frac{dy}{dx}$$

Example 20 — Self Tutor

Find $\dfrac{dy}{dx}$ if: **a** $x^2 + y^3 = 8$ **b** $x + x^2y + y^3 = 100$

a
$$x^2 + y^3 = 8$$
$$\therefore \frac{d}{dx}(x^2) + \frac{d}{dx}(y^3) = \frac{d}{dx}(8)$$
$$\therefore 2x + 3y^2\frac{dy}{dx} = 0$$
$$\therefore \frac{dy}{dx} = \frac{-2x}{3y^2}$$

b
$$x + x^2y + y^3 = 100$$
$$\therefore \frac{d}{dx}(x) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(y^3) = \frac{d}{dx}(100)$$
$$\therefore 1 + \left[2xy + x^2\frac{dy}{dx}\right] + 3y^2\frac{dy}{dx} = 0$$
$$\therefore (x^2 + 3y^2)\frac{dy}{dx} = -1 - 2xy$$
$$\therefore \frac{dy}{dx} = \frac{-1-2xy}{x^2+3y^2}$$

When dealing with implicit relationships, $\dfrac{dy}{dx}$ is usually found in terms of x *and* y. Therefore, to find the gradient of the tangent to the relation at a particular point, we need to know both the x- and y-coordinates of the point.

Example 21 — Self Tutor

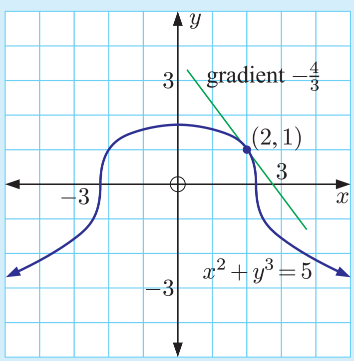
Find the gradient of the tangent to $x^2 + y^3 = 5$ at the point where $x = 2$.

We first find $\dfrac{dy}{dx}$: $\quad 2x + 3y^2\dfrac{dy}{dx} = 0 \quad$ {implicit differentiation}
$$\therefore \frac{dy}{dx} = \frac{-2x}{3y^2}$$

When $x = 2$, $\quad 4 + y^3 = 5$
$$\therefore y = 1$$

$\therefore$ at the point (2, 1), $\dfrac{dy}{dx} = \dfrac{-2(2)}{3(1)^2} = -\frac{4}{3}$

$\therefore$ the gradient of the tangent at $x = 2$ is $-\frac{4}{3}$.



EXERCISE 18H

1 Evaluate, if possible, using l'Hôpital's rule.

a $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$ **b** $\lim\limits_{x\to 0} \dfrac{e^x-1-x}{x^2}$ **c** $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1}$

d $\lim\limits_{x\to\infty} \dfrac{e^x}{x}$ **e** $\lim\limits_{x\to 0^+} x\ln x$ **f** $\lim\limits_{x\to 0} \dfrac{\arctan x}{x}$

g $\lim\limits_{x\to 0} \dfrac{x^2+x}{\sin 2x}$ **h** $\lim\limits_{x\to 0^+} \dfrac{\sin x}{\sqrt{x}}$ **i** $\lim\limits_{x\to 0} \dfrac{x+\sin x}{x-\sin x}$

j $\lim\limits_{x\to 0^+} x^2\ln x$ **k** $\lim\limits_{x\to 0} \dfrac{a^x-b^x}{\sin x}$, where $a, b > 0$ are real constants.

2 Discuss the behaviour of $f(x) = x^2e^{-x}$ as $x\to\infty$.

3 Attempt to find $\lim\limits_{x\to \frac{\pi}{2}^-} \dfrac{\tan x}{\sec x}$ using l'Hôpital's rule.

4 Does $y = \dfrac{\frac{\pi}{2}-\arccos x - x}{x^3}$ have a vertical asymptote when $x=0$? Explain your answer.

5 Consider the function $f(x) = \dfrac{e^x-1}{x^2+x}$.

a State the domain of f. **b** Find $f'(x)$ and state its domain.

c Find the turning point of $y=f(x)$.

d Discuss the behaviour of $f(x)$: **i** as $x\to-\infty$ **ii** near $x=0$.

e Sketch $y=f(x)$, showing the features you have found.

Example 21 Self Tutor

Find $\lim\limits_{x\to 0^+} \dfrac{\ln(\cos 3x)}{\ln(\cos 2x)}$.

$\lim\limits_{x\to 0^+} \ln(\cos 3x) = 0$ and $\lim\limits_{x\to 0^+} \ln(\cos 2x) = 0$, so we can use l'Hôpital's rule.

$$\therefore \lim_{x\to 0^+} \frac{\ln(\cos 3x)}{\ln(\cos 2x)} = \lim_{x\to 0^+} \left(\frac{\frac{-3\sin 3x}{\cos 3x}}{\frac{-2\sin 2x}{\cos 2x}}\right) \qquad \{\text{l'Hôpital's rule}\}$$

$$= \lim_{x\to 0^+} \left(\frac{3\sin 3x\cos 2x}{2\sin 2x\cos 3x}\right)$$

$$= \left(\lim_{x\to 0^+} \frac{\sin 3x}{\sin 2x}\right) \times \left(\lim_{x\to 0^+} \frac{3\cos 2x}{2\cos 3x}\right)$$

$$= \left(\lim_{x\to 0^+} \frac{\sin 3x}{\sin 2x}\right) \times \frac{3}{2}$$

$$= \lim_{x\to 0^+} \frac{\frac{\sin 3x}{3x}}{\frac{\sin 2x}{2x}} \times \frac{9}{4}$$

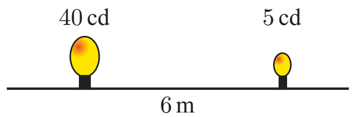
$$= \frac{\lim\limits_{x\to 0^+} \frac{\sin 3x}{3x}}{\lim\limits_{x\to 0^+} \frac{\sin 2x}{2x}} \times \frac{9}{4}$$

$$= \frac{9}{4} \qquad \{\text{Fundamental Trigonometric Limit}\}$$

25 A mosquito flying with position $M(x, y, z)$ is repelled by scent emitted from the origin O. At time t seconds, the coordinates of the mosquito are given by $x(t) = 3 - t^2$, $y(t) = 2 + \sqrt{t}$, and $z(t) = 2 - \sqrt{t}$, where all distance units are metres.

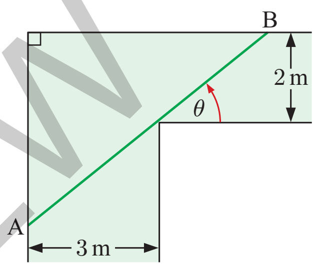
- **a** Show that if the mosquito is D m from the origin at time t, then $D^2 = t^4 - 6t^2 + 2t + 17$.
- **b** *Hence* find the closest the mosquito came to the repellent.

26 Two lamps with luminous intensities 40 candelas and 5 candelas are placed 6 m apart. The intensity of illumination from each lamp is directly proportional to the power of the lamp, and inversely proportional to the square of the distance from the lamp. Find the darkest point on the line segment joining the two lamps.



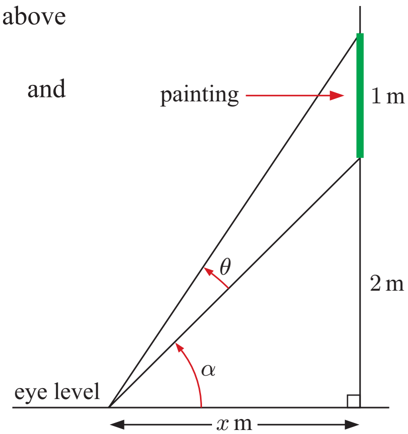
27 Two corridors meet at right angles and are 2 m and 3 m wide respectively. [AB] is a thin metal tube which must be kept horizontal and cannot be bent as it moves around the corner from one corridor to the other.

- **a** Show that the length AB is given by $L = 3\sec\theta + 2\,\text{cosec}\,\theta$.
- **b** Show that $\dfrac{dL}{d\theta} = 0$ when $\theta = \arctan\left(\sqrt[3]{\tfrac{2}{3}}\right) \approx 41.1°$.
- **c** Find L when $\theta = \arctan\left(\sqrt[3]{\tfrac{2}{3}}\right)$, and comment on the significance of this value.

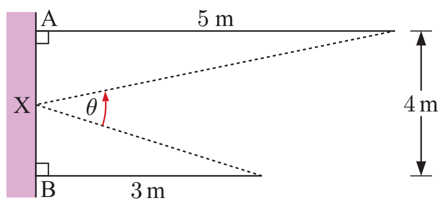


28 Sonia approaches a painting which has its bottom edge 2 m above eye level and its top edge 3 m above eye level.

- **a** Given α and θ as shown in the diagram, find $\tan\alpha$ and $\tan(\alpha + \theta)$.
- **b** Find θ in terms of x only.
- **c** Show that $\dfrac{d\theta}{dx} = \dfrac{2}{x^2+4} - \dfrac{3}{x^2+9}$.
- **d** Find the distance Sonia should stand from the wall in order to maximise her viewing angle of the painting.



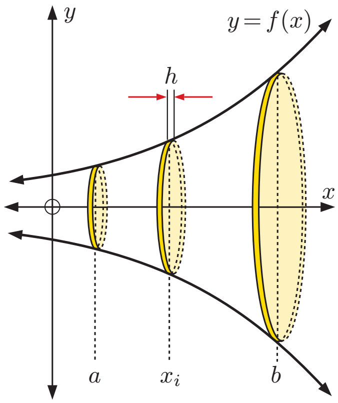
29



How far should X be from A for angle θ to be maximised?

To find the volume of the solid, we can think of it as being made up of n thin cylindrical discs, where the radius of each disc is the value of the function at the left end $x = x_i$, and the height is h. The volume of the ith disc is $\pi[f(x_i)]^2\, h$, so the total volume of the discs is

$$\sum_i \pi[f(x_i)]^2\, h.$$



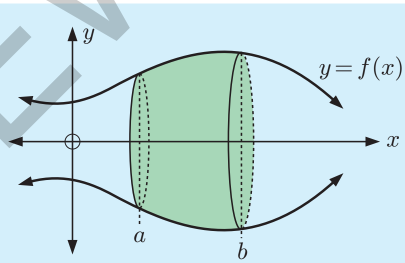
To find the volume *exactly*, we need infinitely many discs. We therefore take the limit as $h \to 0$.

$$\begin{aligned}\text{Volume of revolution} &= \lim_{h\to 0}\sum_i \pi[f(x_i)]^2\, h\\ &= \int_a^b \pi[f(x)]^2\, dx\\ &= \pi\int_a^b [f(x)]^2\, dx\end{aligned}$$

When the region enclosed by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is revolved through 2π about the x-axis to generate a solid, the

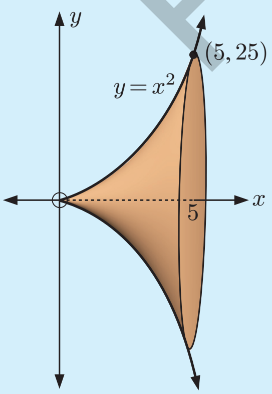
$$\textbf{Volume of revolution} = \pi\int_a^b [f(x)]^2\, dx$$

or $\quad \pi\displaystyle\int_a^b y^2\, dx.$



Example 14 **Self Tutor**

Find the volume of the solid formed when the graph of the function $y = x^2$ for $0 \leqslant x \leqslant 5$ is revolved through 2π about the x-axis.



$$\begin{aligned}\text{Volume of revolution} &= \pi\int_a^b y^2\, dx\\ &= \pi\int_0^5 (x^2)^2\, dx\\ &= \pi\int_0^5 x^4\, dx\\ &= \pi\left[\frac{x^5}{5}\right]_0^5\\ &= \pi(625 - 0)\\ &= 625\pi \text{ units}^3\end{aligned}$$

GRAPHICS CALCULATOR INSTRUCTIONS

A MACLAURIN SERIES

Consider a continuous function f which can be differentiated infinitely many times.
The **Maclaurin series expansion** of $f(x)$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!}$$

For any values of x where the series converges, it is an exact representation of the function.

However, if we consider just the first n terms of the Maclaurin series expansion, we are left with an nth order polynomial approximation for the function.

The ***n*th degree Maclaurin polynomial** approximation to f is $M_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)x^k}{k!}$.

Example 1 Self Tutor

a Find the Maclaurin series representation for $f(x) = \frac{1}{2+x}$.

b Hence find $M_4(x)$, the 4th order Maclaurin polynomial approximation for $\frac{1}{2+x}$. Compare $M_4(1)$ with $f(1)$.

a
$$f(x) = \frac{1}{2+x} = (2+x)^{-1}$$
$$\therefore\ f'(x) = (-1)(2+x)^{-2}$$
$$\therefore\ f''(x) = (-1)(-2)(2+x)^{-3}$$
$$\vdots$$
$$f^{(k)}(x) = (-1)^k k!(2+x)^{-(k+1)}$$
$$\therefore\ f^{(k)}(0) = \frac{(-1)^k k!}{2^{k+1}} \quad \text{for all } k \in \mathbb{Z}^+.$$

Since $f(0) = \frac{1}{2}$, the Maclaurin series representation for $f(x)$ is

$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k \cancel{k!}}{\cancel{k!}\,2^{k+1}} x^k$$
$$= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{k+1}} x^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} x^k$$

b The 4th order Maclaurin approximation for $\frac{1}{2+x}$ is

$$M_4(x) = \sum_{k=0}^{4} \frac{(-1)^k}{2^{k+1}} x^k$$
$$= \frac{1}{2} + \frac{(-1)^1}{2^2}x + \frac{(-1)^2}{2^3}x^2 + \frac{(-1)^3}{2^4}x^3 + \frac{(-1)^4}{2^5}x^4$$
$$= \tfrac{1}{2} - \tfrac{1}{4}x + \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 + \tfrac{1}{32}x^4$$

$\therefore\ M_4(1) = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} = \frac{11}{32}$ compared with $f(1) = \frac{1}{3} = \frac{11}{33}$.

4 Let $y(x) = \sum_{n=0}^{\infty} \frac{p(p-1)....(p-n+1)x^n}{n!}$ for $|\,x\,| < 1$. You may assume the series is on this interval.

a Show that for $|\,x\,| < 1$, y satisfies the differential equation $(1+x)\frac{dy}{dx} = py$.

b Find the general solution to this differential equation.

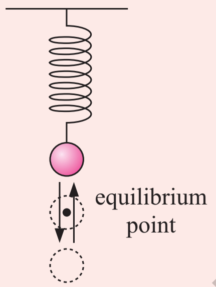
c Hence show that $y(x) = (1+x)^p$ for all $|\,x\,| < 1$.

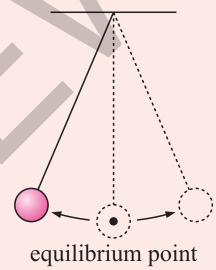
ACTIVITY 1 SIMPLE HARMONIC MOTION

Under **simple harmonic motion**, an object oscillates back and forth about the origin or **equilibrium point**. The object is subject to an acceleration force which is proportional to the object's displacement from the equilibrium point, and in the direction of the equilibrium point.

Simple harmonic motion never exactly occurs in the real world because of friction. However, there are many things that *almost* behave with simple harmonic motion. For example:

- the movement of a mass attached to a spring
- the oscillation of a pendulum.





The acceleration of an object moving with simple harmonic motion is $a = -k^2x$ where x is the displacement from the equilibrium point, and k is a measure of how quickly the object completes a single oscillation.

In the case of a mass on a spring, k is determined by the weight of the object, and the length and stiffness of the spring.

In the case of a pendulum, k is determined by the weight on its end, the length of the pendulum, and its maximum amplitude from the equilibrium point.

From our study of kinematics, we know that the acceleration of the object is $a = \frac{d^2x}{dt^2}$.

We can therefore describe simple harmonic motion by the differential equation $\frac{d^2x}{dt^2} = -k^2x$.

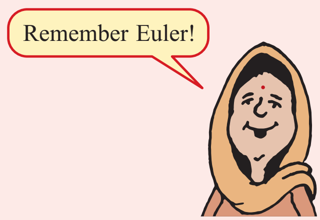
In this Investigation we derive the solution to this differential equation using two different methods.

What to do:

1 Notice that in the differential equation, the second derivative of the function is proportional to the function itself.

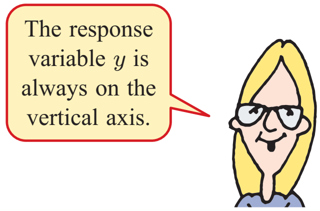
a What function types have we studied with this property?

b How are these function types related?



E THE REGRESSION LINE OF x AGAINST y

In the previous Section, we saw how linear regression can be used to find a linear model for the response variable y in terms of the explanatory variable x.



In these cases, we generally rely on the x-values being more precise than the y-values. This means that either there is less error involved with their measurement, or that there is naturally less variation associated with the x variable. The distance of each data point from the line is measured in the y-direction, so we are associating all of the "error" with the response variable y. However, in some cases the y-values may be more precisely measured.

For example:

- When a student studies for a test, their time spent studying x explains their test score y. However, the test score will be more precisely measured than the amount of time spent studying.
- At a breath testing station, police use a breathalyser to estimate the blood alcohol concentration (BAC) of drivers. If the result x is sufficiently high, the driver is required to take a blood test to establish their actual BAC. The blood test result y is a much more precise measurement.

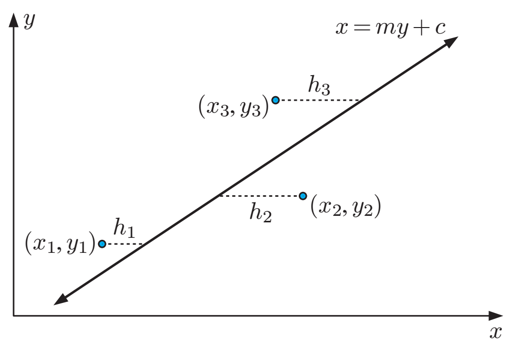
In these scenarios, we consider the regression line of $\boldsymbol{x}$ **against** $\boldsymbol{y}$. This means that we minimise the *horizontal* distances of points from the line, so all of the "error" is associated with the explanatory variable x.

We consider a line of the form $x = my + c$, and choose the constants m and c to minimise $H = \sum_{i=1}^{n} h_i^2$.

It can be shown that $m = \dfrac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sum(y_i - \overline{y})^2}$

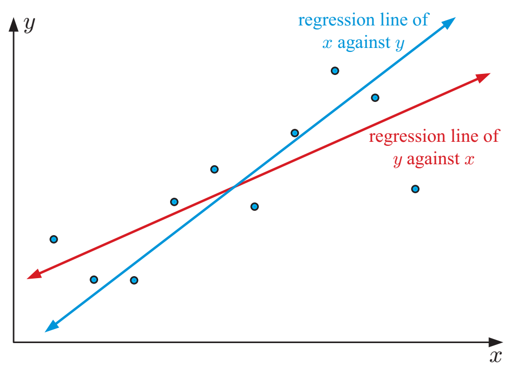
and $c = \overline{x} - m\overline{y}$.

Rearranging the regression line $x = my + c$ into gradient-intercept form gives $y = \dfrac{1}{m}x - \dfrac{c}{m}$.



The gradient of the rearranged line is $\dfrac{1}{m} = \dfrac{\sum(y_i - \overline{y})^2}{\sum(x_i - \overline{x})(y_i - \overline{y})} \neq a$

and the y-intercept is
$$\begin{aligned} -\frac{c}{m} &= \frac{-(\overline{x} - m\overline{y})}{m} \\ &= \frac{m\overline{y} - \overline{x}}{m} \\ &= \overline{y} - \frac{1}{m}\overline{x} \\ &\neq b \qquad \left\{\text{since } \frac{1}{m} \neq a\right\} \end{aligned}$$



So, in general, the regression line of x against y is **not** the same as the regression line of y against x.

5 A die is numbered 1, 1, 2, 3, 3, 3. Let X be the result when the die is rolled once.

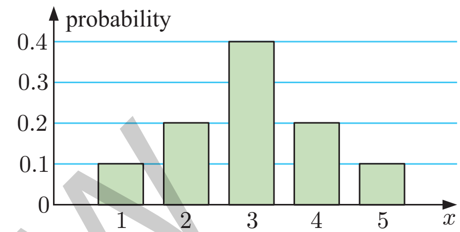
  a Construct the probability distribution for X.
  b Find the mean μ.
  c Find the standard deviation of the distribution.

6 A random variable X has the probability mass function $P(x) = \dfrac{x^2 + x}{20}$, $x = 1, 2, 3$.
For this distribution, calculate the:

  a mode
  b median
  c mean μ
  d standard deviation σ.

7 Use $\sigma^2 = \text{E}[(X - \mu)^2]$ to show that $\sigma^2 = \text{E}[X^2] - (\text{E}(X))^2$.

8 The probability distribution of a random variable X is shown in the graph.

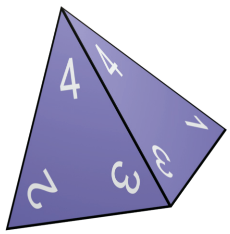


  a Copy and complete:

x_i	1	2	3	4	5
p_i					

  b Find the mean μ and standard deviation σ of the distribution.
  c Find: **i** $\text{P}(\mu - \sigma < x < \mu + \sigma)$ **ii** $\text{P}(\mu - 2\sigma < x < \mu + 2\sigma)$.

9 A tetrahedral die is numbered 1, 2, 3, and 4.
Let X be the number rolled when the die is rolled *once*.
Let Y be the *highest* number rolled when the die is rolled *twice*.



  a Would you expect X or Y to have the greater:
    i mean **ii** standard deviation?
  Explain your answers.
  b Construct probability distributions for X and Y.
  c Which of the variables is a uniform discrete random variable?
  d Calculate the mean and standard deviation of each distribution.

10 Suppose X is a uniform discrete random variable with possible values $X = 1, 2, ..., n$. Show that the variance of X is $\dfrac{n^2 - 1}{12}$.

Hint: $1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

E PROPERTIES OF $aX + b$

If we know the expected value, variance, and standard deviation of a random variable X, we can determine these properties for a related random variable $aX + b$, where a and b are constants.

The following properties of $\text{E}(X)$ are useful:

> If $\text{E}(X)$ is the expected value of the random variable X, then:
>
> **1** $\text{E}(k) = k$ for any constant k
>
> **2** $\text{E}(kX) = k\,\text{E}(X)$ for any constant k
>
> **3** $\text{E}[g(X) + h(X)] = \text{E}[g(X)] + \text{E}[h(X)]$ for functions g and h.

3 Given $X \sim \text{N}(\mu, \sigma^2)$ and $Z \sim \text{N}(0, 1^2)$, determine the values of a and b such that:

- **a** $\text{P}(\mu - \sigma < X < \mu + 2\sigma) = \text{P}(a < Z < b)$
- **b** $\text{P}(\mu - 0.5\sigma < X < \mu) = \text{P}(a < Z < b)$
- **c** $\text{P}(0 \leqslant Z \leqslant 3) = \text{P}(\mu - a\sigma \leqslant X \leqslant \mu + b\sigma)$

4 If $Z \sim \text{N}(0, 1^2)$, find the following probabilities using technology. In each case sketch the region under consideration.

- **a** $\text{P}(0.5 \leqslant Z \leqslant 1)$
- **b** $\text{P}(-0.86 \leqslant Z \leqslant 0.32)$
- **c** $\text{P}(-2.3 \leqslant Z \leqslant 1.5)$
- **d** $\text{P}(Z \leqslant 1.2)$
- **e** $\text{P}(Z \leqslant -0.53)$
- **f** $\text{P}(Z \geqslant 1.3)$
- **g** $\text{P}(Z \geqslant -1.4)$
- **h** $\text{P}(Z > 4)$
- **i** $\text{P}(-0.5 < Z < 0.5)$
- **j** $\text{P}(-1.960 \leqslant Z \leqslant 1.960)$
- **k** $\text{P}(-1.645 \leqslant Z \leqslant 1.645)$
- **l** $\text{P}(|\,Z\,| > 1.645)$

5 **a** Suppose X is normally distributed with mean μ and standard deviation σ.

- **i** Explain why $\text{P}(\mu - 3\sigma < X < \mu + 2\sigma) = \text{P}(-3 < Z < 2)$.
- **ii** Hence find $\text{P}(\mu - 3\sigma < X < \mu + 2\sigma)$.

b For a random variable $X \sim \text{N}(\mu, \sigma^2)$, find:

- **i** $\text{P}(\mu - 2\sigma < X < \mu + 1.5\sigma)$
- **ii** $\text{P}(\mu - 2.5\sigma < X < \mu - 0.5\sigma)$

6 Suppose X is normally distributed with mean $\mu = 58.3$ and standard deviation $\sigma = 8.96$.

a Let the z-score of $x_1 = 50.6$ be z_1 and the z-score of $x_2 = 68.9$ be z_2.

- **i** Calculate z_1 and z_2.
- **ii** Find $\text{P}(z_1 \leqslant Z \leqslant z_2)$.

b Check your answer by calculating $\text{P}(50.6 \leqslant X \leqslant 68.9)$ directly using technology.

HISTORICAL NOTE

The normal distribution has two parameters μ and σ, whereas the standard normal distribution has no parameters. This means that a unique table of probabilities can be constructed for the standard normal distribution.

Before graphics calculators and computer packages, it was impossible to calculate probabilities for a general normal distribution $\text{N}(\mu, \sigma^2)$ directly.

Instead, all data was transformed using the Z-transformation, and the standard normal distribution table was consulted for the required probabilities.

F NORMAL QUANTILES

Consider a population of crabs where the length of a shell, X mm, is normally distributed with mean 70 mm and standard deviation 10 mm.

A biologist wants to protect the population by allowing only the largest 5% of crabs to be harvested. He therefore wants to know what length corresponds to the 95th percentile of crabs.

To answer this question we need to find k such that $\text{P}(X \leqslant k) = 0.95$.

The number k is known as a **quantile**. In this case it is the 95% quantile.