

MYP Mathematics

A concept-based approach

4&5

Standard

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1 Form

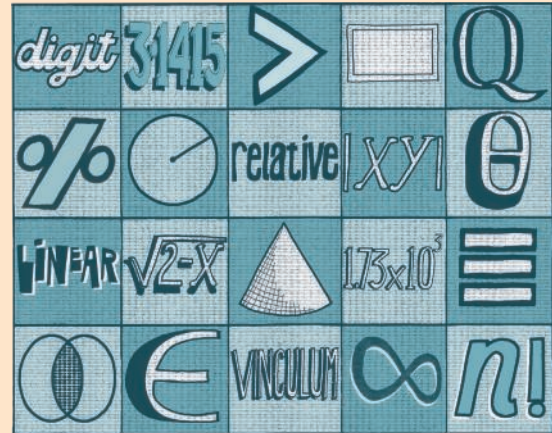
Form is the shape and underlying structure of an entity or piece of work, including its organization, essential nature and external appearance.

The language of mathematics

Mathematics is written using:

- numbers
- words
- symbols
- diagrams
- letters
- shapes

What forms help you recognize mathematics when you see them?

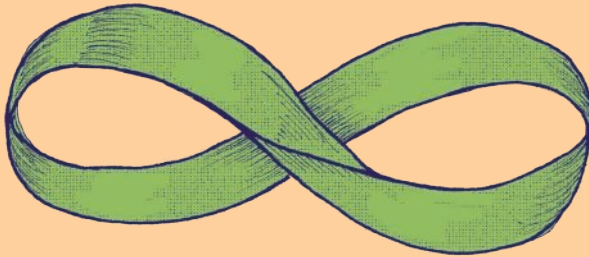


Hilbert's Hotel Infinity

Are there different *forms* of infinity denoting different *sizes* of infinity? David Hilbert's thought experiment imagined a hotel with an infinite number of rooms – with the motto 'always room for one more!'

- What if a guest arrives, but all the infinite number of rooms in the hotel are already occupied? No problem – the guest in room 1 moves to room 2, the guest in room 2 moves to room 3, and so on. Room 1 is now free, and everyone is happy.
- And what if an *infinite* number of guests arrive at the same time but all rooms are occupied? Still no problem – the guest in room 1 moves to room 2, the guest in room 2 moves to room 4, and so on, so that the one in room n moves to room $2n$. All the even-numbered rooms are now occupied, but all the odd numbered rooms are free.

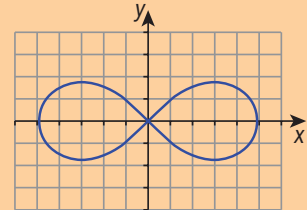
The symbol for infinity is called a lemniscate. The word lemniscate is of Latin origin, and means 'pendant ribbon'.



- Why do you think this form was chosen to symbolize infinity?
- Can you design an alternative symbol that would effectively signify infinity?

This curve is known as the Lemniscate of Bernoulli; on the Cartesian plane its equation is:

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$



- Use a GDC to graph this equation and create a slider for the variable a from -5 to 5. What effect does the variable a have on the graph?

Form and function

In design, form is the shape and appearance of an object. Function is the purpose of the object; does it do the job it was designed for? One definition of a good design is one that balances form and function – it does the job it was designed to do and is also attractive.

- Which do you think is the best chair design?



- Which do you think is the best graphic design?



NASA



1.1

Mathematically speaking

Global context: Scientific and technical innovation

Objectives

- Classifying the different kinds of real numbers
- Representing the different kinds of real numbers
- Drawing and interpreting Venn diagrams to solve problems in real-life contexts
- Applying the language of sets to different areas of mathematics
- Using the language of sets to model real-life problems

Inquiry questions

- F** • What is a set?
- What kinds of sets are there?
- C** • What are set operations?
- How do you represent sets and their operations?
- D** • How useful are sets and Venn diagrams in solving real-life problems?
- How does form influence function?

ATL Communication

Organize and depict information logically



You should already know how to:

describe the different types of real numbers	1 For each number, state which number set(s) it is a member of (e.g. natural numbers, irrational numbers, etc.). a 7 b −7 c 0 d 1.5 e π f −0.1 g $\sqrt{14}$ h $\sqrt{121}$
--	---

F Sets

- What is a set?
- What kinds of sets are there?

Some examples of sets are: the set of students in your class, the set of all your subject teachers this year, the set of your favorite books, the set of parking spaces at your local supermarket.

A **set** is a collection of objects. Each member of a set is called an **element** of the set.

You use capital letters to denote a set, and put the members of the set in curly brackets. You can describe a set using words, or by writing a list of its elements.

Sets described in words:

$A = \{\text{my three favorite colors}\}$

$B = \{\text{vowels in the English alphabet}\}$

$C = \{\text{seven largest moons in the solar system}\}$

$D = \{\text{integers between 1 and 21}\}$

Sets described by listing their elements:

$A = \{\text{blue, orange, green}\}$

$B = \{a, e, i, o, u\}$

$C = \{\text{Ganymede, Titan, Callisto, Io, Earth's Moon, Europa, Triton}\}$

$D = \{2, 3, 4, \dots, 20\}$

The symbol $\in$ means ‘is an element of’.

The symbol $\notin$ means ‘is not an element of’.

For example, if the set $A = \{16, 25, 36, 49, 64\}$, then $49 \in A$ but $81 \notin A$.

The order in which you write the elements of a set is not important.

$B = \{a, e, i, o, u\}$ is the same as $B = \{e, o, u, i, a\}$.

In a class there are ten students aged 16, and four students aged 15.

The set of *students* in the class has 14 elements (the students’ names). The set of their *ages* has only two elements: $\{15, 16\}$, not 16 written ten times and 15 written four times.

The number of elements in a set A is written $n(A)$. It is called the **cardinality** of the set.

...

The three ellipsis dots mean ‘and so on’ or ‘continue the pattern’. It was first used by the ancient Greeks. With sets, the ellipsis is used when writing out all the elements of a set is impractical.

For example if the set $A = \{\text{knife, fork, spoon}\}$ then $n(A) = 3$. If $n(A)$ is a real number, then the set A is a finite set.

You can write common number sets using set notation.

Natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$

Integers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

Positive integers $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$

These are infinite sets, as they contain an infinite number of elements.

Practice 1

1 List the elements of these sets, and find the number of elements in each set.

- a A is the set of the days of the week.
- b B is the set of months of the year not containing the letter r .
- c C is the set of factors of 12.
- d D is the set of positive integers less than 30 that are multiples of 4.

2 Describe each set in words.

- a $J = \{1, 3, 5, 7, 9\}$
- b $K = \{\text{isosceles, equilateral, right-angled, scalene}\}$
- c $L = \{\text{right angle, obtuse, acute, reflex}\}$
- d $M = \{4, 8, 12, 16, \dots, 40\}$

ATL 3 Given the sets $A = \{4, 6, 8, 10\}$, $B = \{1, 8, 27, 64\}$, $C = \{1, 3, 4, 7, 11\}$ and $D = \{0, \pm 1, \pm 2, \pm 3, \pm 4\}$, state whether each statement is true or false. If the statement is false, write the correct statement.

- a $4 \in A$
- b $7 \notin C$
- c $1 \in A$
- d $27 \in A$
- e $8 \in D$
- f $n(C) = n(D)$

Set builder notation

In addition to describing a set in words or by listing its elements it can be described using set builder notation. This uses curly brackets enclosing a variable, a vertical line, and any restrictions on the variable. It can be used for finite or infinite sets. For example, the set $A = \{1, 2, 3, \dots, 9\}$ can be written in set builder notation, which would look and read like this:

$$A = \{x \mid x \in \mathbb{N}, 1 \leq x \leq 9\}$$

A is the set of all values of x such that x is a natural number and x is between 1 and 9 inclusive.

The symbols for numbers sets are often referred to as 'double struck' because they could be produced by double-striking a character on a typewriter, or 'blackboard bold' as they were written with the edge of the chalk on a blackboard instead of the point, to differentiate them from other bold characters.

Tip

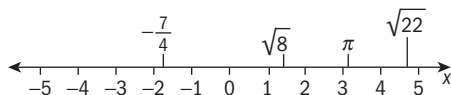
For natural numbers, $1 \leq x \leq 9$ is equivalent to $0 < x < 10$.

Using set builder notation you can define another number set that you need to know: the rational numbers

Rational numbers: $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$

In words, $\mathbb{Q}$ is the set of rational numbers (numbers of the form $\frac{p}{q}$) such that both p and q are integers, and q does not equal 0.

All of the integers, rational numbers and irrational numbers together form the set of real numbers, $\mathbb{R}$. The real numbers can be represented on the real number line.



We don't have a symbol for irrational numbers. They are numbers that cannot be expressed as a fraction, for example: π or $\sqrt{2}$.

Example 1

Write these sets in set builder notation.

a S is the set of real numbers between 0 and 1

b $P = \{2, 3, 5, 7, \dots, 37\}$

c $M = \{2, 4, 6, 8, \dots\}$

a $S = \{x \mid x \in \mathbb{R}, 0 < x < 1\}$

Use the 'less than' symbol because 0 and 1 are not included.

b $P = \{x \mid x \in \text{primes}, 2 \leq x \leq 37\}$

There is no special symbol for prime numbers.

c $M = \{x \mid x = 2n, n \in \mathbb{Z}^+\}$

Even numbers are multiples of 2, so they can be written as $2n$.

Example 2

Write out the elements of each set in list form.

a $E = \{x \mid x \in \mathbb{Z}, -3 < x < 2\}$

b $F = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \right\}$

Describe set G in words.

c $G = \{x \mid x \in \mathbb{R}, 0 < x < 1\}$

a $E = \{-2, -1, 0, 1\}$

-3 and 2 are not included.

b $F = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$

c G is the set of real numbers greater than 0 and less than 1.

There is no first real number greater than 0, or last real number less than 1, so you cannot list the elements of this set.

Practice 2

1 Write out the elements of each set in list form. State whether each set is finite or infinite. If the set is finite, state its cardinality.

a $\{x | x \in \mathbb{Z}, -2 < x < 5\}$

b $\{y | y \in \mathbb{Z}, y > 0\}$

c $\{a | a \in \mathbb{N}, a \text{ is a multiple of } 5\}$

d $\{b | b \in \mathbb{R}, b \text{ is a factor of } 28\}$

e $\{c | c \in \mathbb{N}, c + 3 < 8\}$

f $\{p | p \in \text{primary colors}\}$

2 Write each set using set builder notation.

a $S = \{1, 9, 25, 49, 81, \dots\}$

b $T = \{\dots, -10, 0, 10, 20, \dots\}$

c U is the set of real numbers between 1 and 2, including 2.

d V is the set of rational numbers between 0 and 1.

Problem solving

3 Write in set builder notation:

a The set of odd numbers.

b The set of multiples of 3.

c $W = \{1, 2, 4, 8, 16, 32\}$

The **universal set**, U , is the set that contains all sets being considered.

The **empty set**, $\{\}$ or $\emptyset$, is the set with no elements, so $n(\emptyset) = 0$.

For example, for the set W of winter months and set S of summer months, the universal set U is the set of months of the year.

From the universal set $U = \{\text{yellow, red, blue}\}$ you can make these sets:

$J = \emptyset$

$P = \{\text{yellow, red}\}$

$K = \{\text{yellow}\}$

$S = \{\text{yellow, blue}\}$

$L = \{\text{red}\}$

$T = \{\text{red, blue}\}$

$M = \{\text{blue}\}$

$U = \{\text{yellow, red, blue}\}$

All these sets, which can be made from the elements of U , including the empty set and U itself, are called subsets of U .

The set A is a **subset** of a set B if every element in A is also in B . The symbol for subset is $\subseteq$. Written in mathematical form:
if for all $x \in A \Rightarrow x \in B$, then $A \subseteq B$.

The **empty set** is a subset of any set. So, for any set A , $\emptyset \in A$.

Every set is a subset of itself.

In examinations, the universal set will be given to you in the question, when necessary.

' $x \in A \Rightarrow x \in B$, then $A \subseteq B$ ' can be read as 'If for all elements in A the elements are also in B , then A is a subset of B .'

Two sets are equal if they contain exactly the same elements.

Some further examples of subsets are:

- If $A = \{1, 5\}$ and $B = \{1, 2, 3, 4, 5\}$, then $A \subseteq B$.
- If $C = \{3^n \mid n \in \mathbb{N}\}$ and $D = \{1, 3, 9, 27\}$, then $D \subseteq C$, because $D = \{3^0, 3^1, 3^2, 3^3\}$
- If $E = \{x \mid x^2 = 1\}$ and $F = \{-1, 1\}$, then $E \subseteq F$ and $F \subseteq E$ are both true. In other words, $E = F$.

You can always use the symbol $\subseteq$ to denote subset. If two sets are not equal, (when there is at least one element of A that is not in B) then you can use the symbol $\subset$ without a line underneath it. $A \subset B$ means A is a **proper subset** of B .

If $A = B$, then A is an improper subset of B , or B is an improper subset of A , and in this case you use the symbol $\subseteq$.

The symbol $\subseteq$ and $\subset$ are similar in the way they work to $\leq$ and $<$. If in doubt use $\subseteq$ as it can be used for an improper or proper subset.

Exploration 1

- 1 Consider a set containing two elements, for example, $A = \{1, 2\}$. Write down all of its subsets.
- 2 Now consider a set containing three elements, for example $B = \{1, 2, 3\}$. Write down all of its subsets.
- 3 Do the same again for a set containing four elements.
- 4 Based on your findings, suggest a rule for determining the number of subsets of a set with n elements.
- 5 Test your rule on a set containing five elements.

The set itself and the empty set are subsets of any given set.

Reflect and discuss 1

- For the set $A = \{1, 2\}$, is every element of $\emptyset$ a member of A ?

Is there any element of $\emptyset$ that is not in A ?

Use your answers to explain why $\emptyset$ is a subset of A , and of any set.

A generalization or general rule is 'a general statement made on the basis of specific examples'.

- What specific examples did you use in Exploration 1?
- Compare the rule you found in Exploration 1 with others in your class. Did you all get the same result? Is that enough to say that this rule is always true for any number of elements in the original set?
- Why should you be cautious when generalizing?

Later in your mathematics course you will learn about the key concept of Logic and the related concept of Justification. Until you have justified your general statement, you cannot be certain it will always be true.

Practice 3

- 1 Determine whether or not these pairs of sets are equal.
 - a $\{1, 2, 3\}; \{2, 3, 1\}$
 - b $\{1, 2, 3, 5\}; \{\text{prime numbers less than } 6\}$
 - c $\{16, 17, 18, 19, \dots\}; \{\text{rational numbers greater than } 15\}$
- 2 Determine whether these statements are true or false. If the statement is false, give a reason why.
 - a $2 \in \{2, 3, 4\}$
 - b $\{2\} \subseteq \{2, 3, 4\}$
 - c $\{3\} \in \{2, 3, 4\}$
 - d $4 \subseteq \{2, 3, 4\}$
 - e Given $A = \{x \mid x \in \mathbb{Z}, 1 < x < 5\}$ and $B = \{2, 3, 4\}$, then $A = B$.

Problem solving

- 3 Determine if the statements **a** to **d** below are true or false. For those that are false, give a counter-example (find an example that makes the statement false). You may find drawing a diagram helpful.
 - a If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
 - b If $R \subseteq S$ then $S \subseteq R$.
 - c If $p \in P$ and $P \subseteq Q$, then $p \in Q$.
 - d If $A = B$, then $B \subseteq A$.
- 4 Write a set that has 32 subsets.
- 5 Determine whether or not a set can have exactly 24 subsets.



Set operations

- What are set operations?
- How do you represent sets and their operations?

Just as you use arithmetic operations to manipulate numbers, sets also have their own operations.

Union and intersection of sets

- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, the set that contains all the elements that are in both A and B without repeating any of them is $C = \{1, 2, 3, 4, 5\}$. Set C is the **union** of sets A and B , and is written $A \cup B$.

The set that contains the elements that both sets have in common is $D = \{2, 3\}$. D is the **intersection** of sets A and B , and is written $A \cap B$.

The **complement** of a set A is denoted by A' , and described in set notation as $A' = \{x \mid x \in U, x \notin A\}$.

For the universal set $U = \{1, 2, 3, 4, 5, 6\}$, and set $E = \{4, 6\}$, the complement of the set E is made up of the elements in the universal set that are *not* in set E . So $E' = \{1, 2, 3, 5\}$.

Example 3

If $U = \{1, 2, 3, \dots, 10\}$, $A = \{2, 4, 6, 8\}$, $B = \{2, 3, 5, 7\}$ and $C = \{1, 5, 9\}$, find:

a $A \cap C$

b $A \cup B'$

c $(A' \cap B)'$

d $(A \cup B \cup C)'$

a $A \cap C = \emptyset$

A and C have no elements in common.

b $B' = \{1, 4, 6, 8, 9, 10\}$

First find B' .

$A \cup B' = \{1, 2, 4, 6, 8, 9, 10\}$

c $A' = \{1, 3, 5, 7, 9, 10\}$

$A' \cap B = \{3, 5, 7\}$

First find $A' \cap B$.

$(A' \cap B)'$ is made up of the elements in U that are not in $A' \cap B$.

$(A' \cap B)' = \{1, 2, 4, 6, 8, 9, 10\}$

d $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$(A \cup B \cup C)' = \{10\}$

ATL

Practice 4

1 If $U = \{x | x \in \mathbb{Z}^+, 1 \leq x \leq 20\}$, $A = \{2, 4, 8, \dots, 20\}$, $B = \{1, 3, 5, \dots, 19\}$ and $C = \{x | x \in \text{primes}, 1 \leq x \leq 20\}$, find:

a $A \cap B$

b $A \cup B$

c $A' \cap C$

d $(A \cap C)'$

e $(A' \cup B)'$

2 If $U = \mathbb{Z}$, $R = \{x | x \in \mathbb{Z}^+, x < 10\}$, $S = \{x | -5 < x < 5\}$ and $T = \{x | x \in \mathbb{N}; x \leq 15\}$, find:

a $R \cap S$

b $R \cup T$

c $R' \cap T$

d $S' \cap T$

e $(R \cap S)' \cap T$

Problem solving

3 The universal set is $U = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$. Write three subsets F , G and H such that:

• $n(F \cap G) = 2$

• $F \cap H = \emptyset$

• $G' = \{x | x \text{ is odd}, 11 \leq x \leq 19\}$

Reflect and discuss 2

For any two sets, A and B , explain why the intersection $A \cap B$ is a subset of their union $A \cup B$.

Tip

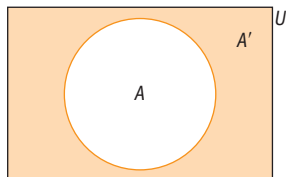
Try with any two sets A and B . Can you generalize your result?

Venn diagrams

In a Venn diagram, a rectangle represents the universal set.

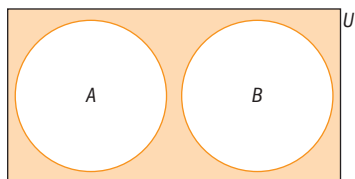
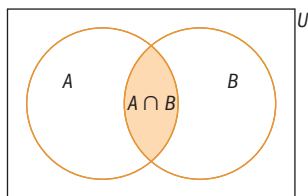
Circles represent subsets of the universal set.

It is important that you use the forms of representation correctly – remembering to include the universal set in your Venn diagrams, for example, or using curly brackets when listing the elements of a set.



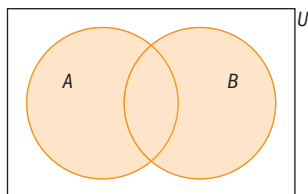
Venn diagram for a set A and its complement A'

Venn diagram of the intersection of the two sets A and B : $A \cap B$



This Venn diagram shows that the intersection of the two sets A and B is the empty set: $A \cap B = \emptyset$. Sets A and B are **disjoint**.

This one shows the union of two sets A and B . The unshaded area outside the circles is therefore the complement of the union of A and B : $(A \cup B)'$.



Exploration 2

- 1 Draw two Venn diagrams: in the first one shade the region that represents the set $A' \cap B'$; in the second diagram shade the region that represents the set $(A \cup B)'$. Describe the relationship between these two sets mathematically.
- 2 Draw two Venn diagrams: in the first one shade the region that represents the set $(A \cap B)'$; in the second diagram shade the region that represents the set $A' \cup B'$. Describe the relationship between these two sets mathematically.
- 3 Draw a Venn diagram to show that for any two sets, A and B , the intersection $A \cap B$ is a subset of their union $A \cup B$. Which do you think was easier – proving this using reasoning in Reflect and discuss 2, or by drawing a diagram?

Reflect and discuss 3

- How many different ways have you represented the set $A' \cup B'$ in Exploration 2?
- What can you say about $A' \cup B'$ and $(A \cap B)'$?
- How has representing sets using a Venn diagram helped you discover new information about sets?

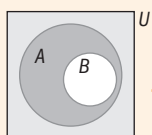
The relationships you have discovered so far by using Venn diagrams are called De Morgan's Laws, named after the British mathematician Augustus De Morgan. They are used extensively in the fields of circuitry and electronics, as well as in the field of logic.

Example 4

Draw Venn diagrams to represent these sets or relationship between sets.

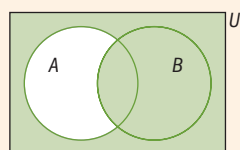
a $B \subseteq A$ **b** $A' \cup B$ **c** $A \cap B'$ **d** $A \cap B \cap C$ **e** $(A \cap B) \cup C$

a $B \subseteq A$



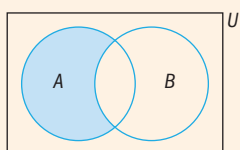
Draw a rectangle representing the universal set. Set B is completely contained in A .

b $A' \cup B$



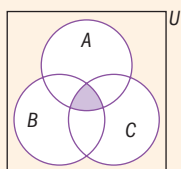
Shade the area that contains everything that is not in A , together with all of B .

c $A \cap B'$



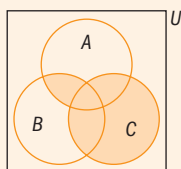
The shaded area represents what A and the complement of B have in common.

d $A \cap B \cap C$



The shaded area shows the intersection of all three sets.

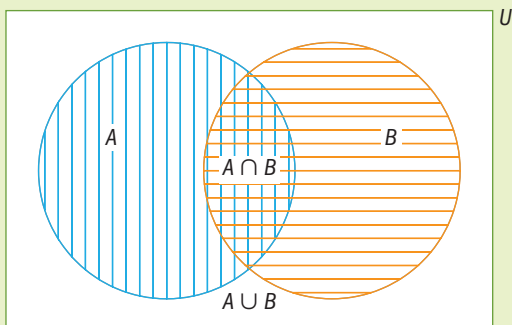
e $(A \cap B) \cup C$



Shade the intersection of A and B first, then shade all of C .

Tip

When drawing Venn diagrams it can be useful to shade one set using vertical lines and the other using horizontal lines. The union will be the total area shaded and the intersection of the sets is where the lines cross.



ATL

Practice 5

1 Draw Venn diagrams to represent each set.

a $(A \cap B)'$

b $A \cup (A \cap B)$

c $(A' \cap B)'$

d $(A' \cup B')'$

e $A \cup (B \cap C)$

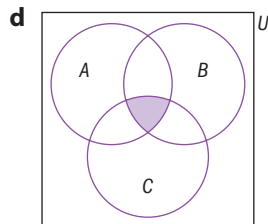
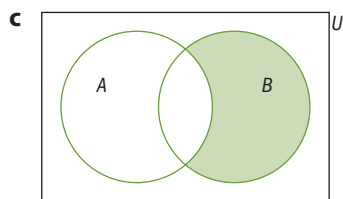
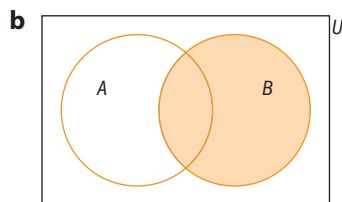
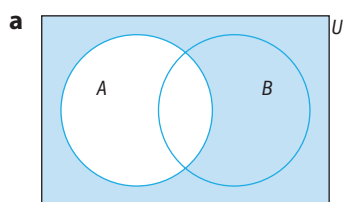
f $(A \cap B)' \cup C$

g $A \cap (B \cup C)'$

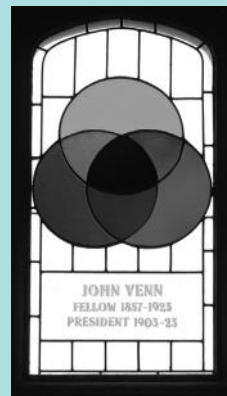
h $(A \cap B) \cup (A \cap C) \cup (B \cap C)$

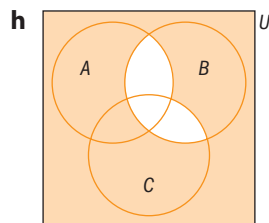
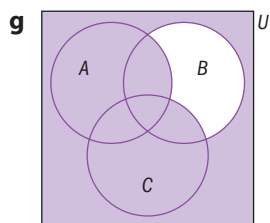
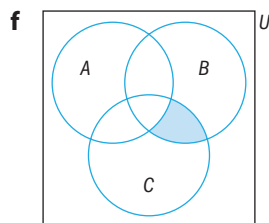
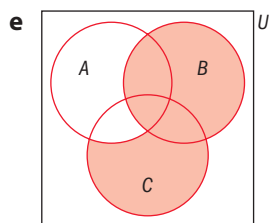
Problem solving

2 Write the set that the shaded part of each Venn diagram represents.



Venn diagrams are named after an English mathematician, John Venn. This stained glass window in Cambridge University commemorates his achievements.





You are already familiar with the properties of real numbers under the binary operations of addition and multiplication. In the following exploration you will determine which of these properties are also valid under the set operations of intersection and union.

Exploration 3

- 1** By drawing a Venn diagram for each side of the equals sign in the statements below, show that the following properties hold for set operations:

a $A \cap B = B \cap A$

b $A \cup B = B \cup A$

Commutative laws

c $A \cap (B \cap C) = (A \cap B) \cap C$

d $A \cup (B \cup C) = (A \cup B) \cup C$

Associative laws

e $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

f $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Distributive laws

- 2** The additive identity of the real numbers is 0 since $n + 0 = 0 + n = n$ and the inverse of n is $-n$ since $n + (-n) = (-n) + n = 0$.

The multiplicative identity of m is 1 since $m \times 1 = 1 \times m = m$ and the inverse of m is $\frac{1}{m}$ since $m \times \frac{1}{m} = \frac{1}{m} \times m = 1$ ($m \neq 0$).

Determine whether or not there is an identity and inverse for the operations $\cup$ and $\cap$.

Find sets B and C such that $A \cup B = B \cup A = A$ and $A \cap C = C \cap A = A$. B and C are the two identities.

Find if sets E and F exist such that $A \cup E = E \cup A = B$ and $A \cap F = F \cap A = C$. E and F , if they exist, are the inverses.

- 3** Summarize the properties of the set operations union and intersection.

D Modelling real-life problems using Venn diagrams

- How useful are sets and Venn diagrams in solving real-life problems?
- How does form influence function?

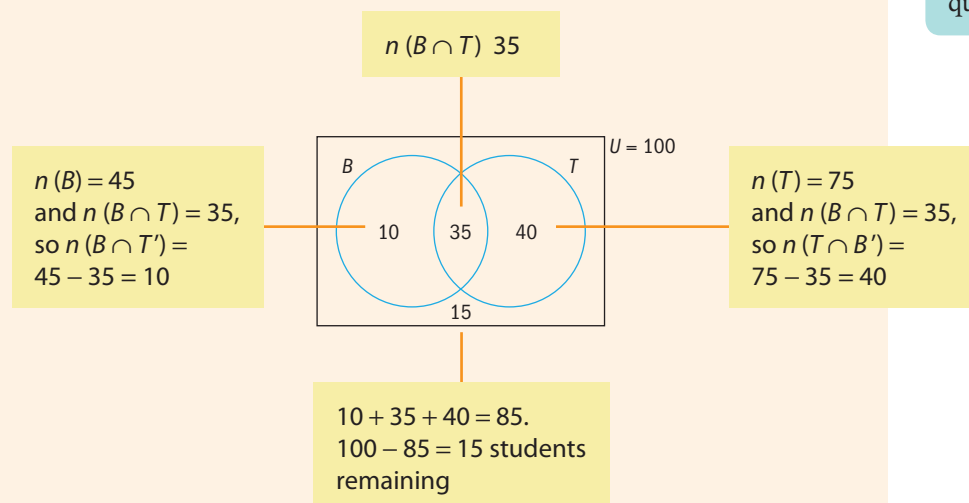
Venn diagrams are used to model and solve problems in fields such as market research, biology, and social science, where ‘overlapping’ information needs to be sorted.

Example 5

A market research company surveys 100 students and learns that 75 of them own a television (T) and 45 own a bicycle (B). 35 students own both a television and a bicycle. Draw a Venn diagram to show this information.

- Find how many students own either a television or a bicycle, or both.
- Find how many own neither a television nor a bicycle.
- Explain how your diagram shows that 50 students own either a television or a bicycle, but not both.

Set theory is used by online shopping search engines. Data on products for sale is split into sets so that your search query will get a response more quickly.



- The number of students who own either a television or a bicycle, or both is $n(B \cup T) = 85$.
- The number of students who own neither a television nor a bicycle is $n(B \cup T)' = 15$.
- The number of students who own either a television or a bicycle but not both can be seen as the union of the two sets less the intersection, or $10 + 40 = 50$.

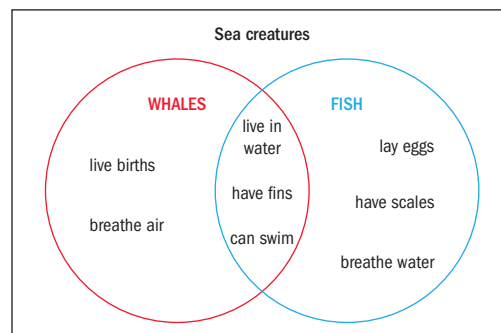
Practice 6

Objective: C. Communicating

v. organize information using a logical structure

These questions encourage students to use Venn diagrams (and a table in Question 1) to organize information logically. The students should be able to draw and interpret Venn diagrams.

- 1 **a i** For the Venn diagram, describe the characteristics of only whales, of only fish, and those of both whales and fish.
- ii** Explain how the diagram has helped you with your descriptions and whether a different form could be more useful.
- b i** Below is a table describing some characteristics of whales, fish, and shrimp. Create a Venn diagram that illustrates these characteristics. Make sure to choose an appropriate universal set.



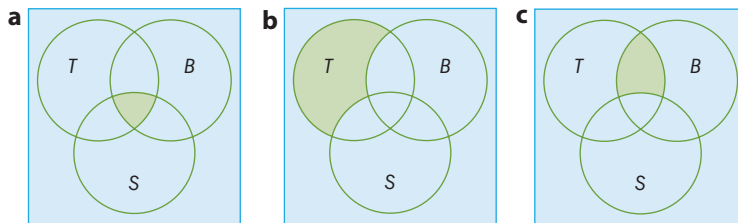
	live births	fins	have scales	lay eggs	live in water	breathe air	breathe water	have legs
whales	✓	✓			✓	✓		
fish		✓	✓	✓	✓		✓	
shrimp				✓	✓		✓	✓

- ii** Explain whether the table form or the Venn diagram form is best to illustrate the given information.
- 2 In a year group of 50 students, 18 are enrolled in Music, 26 in Art, and 2 in both Music and Art. By drawing a Venn diagram, determine how many students are not enrolled in either Music or Art.
- 3 In a school of 350 students, 75 are involved in community service projects, 205 in athletics, and 62 are involved in both. By drawing a Venn diagram, determine how many students are involved in either one of these two activities.
- 4 In a school cafeteria at lunchtime, 93 students chose a soft drink and 47 students chose bottled water. 25 students chose both drinks. If each student chose one of these drinks, determine the total number of students in the cafeteria by drawing a Venn diagram.
- 5 Of 150 new university students, 85 signed up for Mathematics and 70 for Physics, while 50 signed up for both subjects. By drawing a Venn diagram, determine how many students signed up for:
 - a** only Mathematics
 - b** only Physics
 - c** neither Mathematics nor Physics.
- 6 There are 30 students enrolled in three different school clubs: chess, archery and cookery. Of these, 5 students are in all three clubs, 6 of them are only in the cookery club, 2 are in chess and cookery but not archery, 15 belong to cookery in total, 2 are only in chess, and 3 are only in archery. By drawing a Venn diagram, determine how many students are in:
 - a** the archery and cookery clubs only
 - b** the chess club.

- 7 In a class of 32 students, 5 live in the school town and travel to school by bus, and they have school lunches. 3 live in the school town and travel to school by bus, but do not have school lunches. 9 students do not live in the school town, do not travel to school by bus, and do not have school lunches. 11 students live in the school town and have school lunches. A total of 16 students live in the school town. 9 students travel to school by bus and have school lunches. 13 students travel to school by bus. By drawing a Venn diagram, determine how many students in total have school lunches.

Problem solving

- 8 The Venn diagrams below represent participants in an after school sports program. The students can choose to select table tennis (T), basketball (B), or squash (S).



Write in words what each diagram represents.

Exploration 4

- 1 $n(U) = 105$, $n(A) = 37$, and $n(B) = 84$. All of the elements of U are either in A or B or both.
 - a Draw a Venn diagram showing A , B and U .
 - b Find a rule connecting the number of elements in $A \cup B$ with the number of elements in sets A , B , and the intersection of A and B .
 - c Justify your rule.
- 2 Apply your rule to the following situation. Your student council held a vote on which charity to support. A mobile library received 47 votes, and a neighborhood watch committee received 36 votes. 24 students voted for both charities. Determine how many students actually voted.

Reflect and discuss 4

In Exploration 4:

- How have you used sets to model the situation in step 2?
- Why was it easier to consider sets of students rather than considering the students individually?
- What set operation models the fact that some students voted for both charities?

Reflect and discuss 5

Working with one or more of your classmates, discuss how Venn diagrams can be used to organize and categorize some of the following situations: your school timetable, your household chores, your homework for different subjects, your test schedules. Create two Venn diagrams with at least three circles to illustrate the different situations you have chosen.

Summary

A **set** is a collection of objects. Each member of a set is called an **element** of the set.

The symbol $\in$ means 'is an element of'.

The symbol $\notin$ means 'is not an element of'.

The number of elements in a set A is written $n(A)$. It is called the **cardinality** of the set.

The **universal set**, U , is the set that contains all sets being considered.

The **empty set**, $\{ \}$ or $\emptyset$, is the set with no elements, so $n(\emptyset) = 0$.

The set A is a **subset** of a set B if every element in A is also in B . The symbol for subset is $\subseteq$.

Written in mathematical form, if for all $x \in A \Rightarrow x \in B$, then $A \subseteq B$.

The **empty set** is a subset of any set. So, for any set A , $\emptyset \in A$.

Every set is a subset of itself.

Two sets are equal if they contain exactly the same elements.

Union and intersection of sets

- Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

The complement of a set A is denoted by A' , and described in set notation as $A' = \{x \mid x \in U, x \notin A\}$.

Mixed practice

- Write down** the elements of each set in list form.
 - A is the set of names of all continents.
 - B is the set of names of the three tallest mountains in the world.
 - C is the set of names of the three tallest buildings in the world.
 - D is the set of positive integers between 20 and 50, including 20 and 50, that are multiples of 5.
- State** whether each set is finite or infinite. If the set is finite, **state** its cardinality.
 - $\{x \mid x \in \mathbb{N}, x \text{ is a multiple of } 4\}$
 - $\{x \mid x \in \mathbb{R}, -2 \leq x \leq 5\}$
 - $\{y \mid y \in \mathbb{Z}, -3 < y \leq 2\}$
 - $\{b \mid b \in \mathbb{N}, b < 7\}$
- Write down** these sets using set builder notation:
 - F is the set of integers greater than zero
 - $G = \{3, 6, 9, \dots, 21\}$
 - $H = \{1, 4, 9, 16, \dots\}$
 - $J = \{1, 8, 27, \dots, 1000\}$
- $U = \{x \mid x \in \mathbb{N}\}$,
 $A = \{x \mid x \in \mathbb{N}, x \text{ is a multiple of } 7\}$,
 $B = \{1, 3, 5, 7, 9, \dots, 19\}$ and
 $C = \{x \mid x \in \text{primes } 5 < x \leq 20\}$.
Find:
 - $A \cap B$
 - $A' \cap C$
 - $A \cap B'$
 - $(A \cap C)'$
- Create** Venn diagrams to represent the sets:
 - $A \cap B'$
 - $A' \cup B$

- 6 The table shows the facilities at three hotels in a resort.

	Hotel A	Hotel B	Hotel C
Pool	✓		✓
Bar	✓	✓	✓
Jacuzzi		✓	
Sauna	✓		
Tennis		✓	✓
Gym			✓

Draw a Venn diagram to represent this information.

- 7 **Create** a Venn diagram to illustrate the relationships of the following number sets:

$\mathbb{R}$ = {real numbers}, $\mathbb{Z}$ = {integers},
 $\mathbb{Q}$ = {rational numbers}, I = {irrational numbers}, $\mathbb{N}$ = {natural numbers}, and
 P = {prime numbers}.

- 8 **Create** a Venn diagram illustrating the relationships of the following geometric figures:
 U = {all quadrilaterals}, P = {parallelograms},
 R = {rectangles}, S = {squares}, K = {kites},
 T = {trapezoids}.

- 9 If $U = \{-10, -9, \dots, 9, 10\}$, $A = \{0, 1, 2, \dots, 9\}$,
 $B = \{-9, -8, \dots, 0\}$ and $C = \{-5, -4, \dots, 4, 5\}$,
list the elements of the following sets:

- a $A \cap B$ b $(B \cup C)'$
c $(A \cup B) \cap C$ d $A' \cap (B \cup C)$
e $(A \cap B) \cup (A \cap C)$

- 10 **Determine** whether the following statements are true or false. If false, explain why.

- a $0 \in \mathbb{Q}$
b $\{\text{primes}\} \subseteq \{\text{odd integers}\}$
c If $U = \mathbb{N}$, then $(\mathbb{Z}^+ \cap \mathbb{N})' = \{0\}$
d $2 \subseteq \{\text{primes}\}$

Problem solving

- 11 In a school sports day, medals were awarded as follows: 36 in running, 12 in high jump, and 18 in discus. The medals were awarded to a total of 45 students, and only 4 students received medals in all three events. **Determine** how many students received medals in exactly two out of three events.

- 12 A survey of 39 university students found:

- 10 worked part-time while studying
- 18 received financial help from home
- 19 withdrew money from their savings as needed
- 2 financed themselves from all three sources
- 12 received financial help from home only
- 5 received financial help from home and withdrew money from savings
- 6 financed themselves only by working part-time and withdrawing money from savings.

Determine how many students:

- a did not finance themselves using any of the three resources
b worked part-time and received money from home
c received financial help from home and withdrew money from their savings
d financed themselves using only one of the three ways surveyed.

Review in context

Scientific and technical innovation

- 1 Below is a Venn diagram used by medical researchers showing the genes associated with different brain diseases.

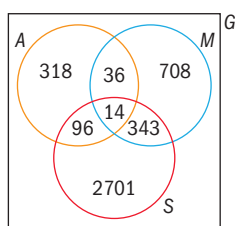
The sets represent the number of genes associated with:

$A = \{\text{Alzheimer's disease}\}$

$M = \{\text{multiple sclerosis}\}$

$S = \{\text{stroke}\}$

$G = \{\text{brain diseases}\}$



Using this diagram:

- Calculate** how many genes in total are associated with each of the three diseases: Alzheimer's, multiple sclerosis, and stroke.
- State** how many genes all three diseases share in common.
- Explain** how this diagram might be useful to medical researchers.

Problem solving

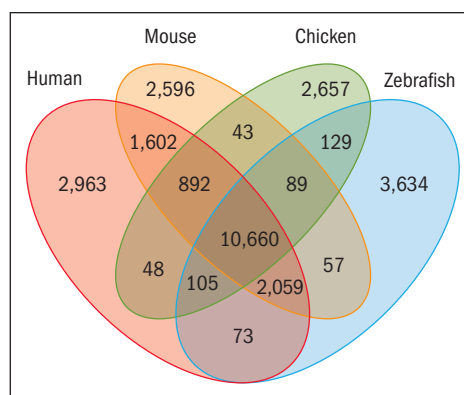
- 2 Venn diagrams are widely used to show 'overlapping' concepts. Here are two fields from Sociology and Environmental Science where a Venn diagram can be used to illustrate the interrelation of key concepts.

In each example, **identify** sets and **draw** a Venn diagram to illustrate the information given.

- According to Plato there are many propositions. Some of these are true and some are beliefs (some may be neither). Only true beliefs can be justified and those he defines as knowledge.
- There are three main types of development: environmental, social and economic. Development that is both social and economic is equitable, that which is economic and environmental is viable, and that which is environmental and social is bearable. Development that is all three of these is sustainable.

- 3 Scientists have studied the genome of a number of organisms. The genome contains all the information used to build and maintain that organism. For medical researchers, knowing that different species share particular genes will enable them to unlock the secrets of countless diseases.

The Venn diagram shows the genes of four species: human, mouse, chicken, and zebrafish.



Using the diagram:

- Determine** how many genes are shared by all four species.
- Determine** how many genes does a human have in total.
- Determine** how many genes are shared by human and mouse, by human and chicken, and by human and zebrafish.
- Determine** the percentage of genes shared by the human and each of these three species.
- If you were a medical scientist and you wanted to conduct research into a species that was genetically closest to the human, which of the three – mouse, chicken or zebrafish – would you choose?